COMPARING TOPOLOGICAL AND ARITHMETIC KAC-MOODY GROUPS

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Abstract. A map $BK(\overline{\mathbb{F}}_p) \to BK$ from the classifying space of the discrete Kac-Moody group over the algebraic closure of the field with p elements to the classifying space of a complex topological Kac-Moody group is constructed. This map induces a $\mathbb{F}_q$-homology isomorphism for all primes $q \neq p$ and generalizes a map for Lie group classifying spaces constructed in [22]. As an application, unstable Adams operations, ψ^k, for general Kac-Moody groups are constructed which are compatible with the Frobenius. Interestingly, homotopy fixed points $BK^{h\psi^k}$ do not agree with $BK(\mathbb{F}_{p^k})$ upon localization with respect to $\mathbb{F}_q$-homology. These results rely on new homotopy decompositions for certain infinite "unipotent" subgroups of $K(R)$ in terms of finite dimensional unipotent groups.

Contents

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1. Introduction

This paper studies Kac-Moody groups, a generalization of Lie groups though typically infinite dimensional in nature, by using homotopy theory. More specifically, the classifying spaces of Kac-Moody groups and important subgroups are investigated and in this sense, this is a paper in homotopical Kac-Moody group theory. Homotopical group theory (c.f. [27]) simply means to study groups via their classifying spaces. We will be primarily concerned with discrete Kac-Moody groups over finite fields $\mathbb{F}_{p^k}$ or their algebraic closure $\overline{\mathbb{F}}_p$ and *topological* Kac-Moody groups over the complex numbers.

Topological Kac-Moody groups provide interesting and tractable examples of "infinite dimensional Lie groups." An array of new techniques have been developed to study their algebra, geometry, topology, and representation theory (c.f. [29–31, 36, 37]). Parallels to the Lie groups abound including the relevance of Kac-Moody groups to topics across the mathematical spectrum from combinatorics to mathematical physics. Kumar [36] provides a well-written, comprehensive and modern account of this material.

Discrete Kac-Moody groups over $\mathbb{F}_{p^k}$ or $\overline{\mathbb{F}}_p$ have a more algebraic and number theoretic character. For instance, $G(\mathbb{F}_{p^k}[t, t^{-1}])$ for G a semi-simple algebraic group of Lie type provides examples arithmetic groups that are affine Kac-Moody groups over $\mathbb{F}_{p^k}$. Rèmy [42] argues that discrete Kac-Moody groups over finite fields should be considered generalizations of certain S-arithmetic groups. Moreover, discrete Kac-Moody groups over finite fields are finitely generated and provide an important source of examples (c.f. [19]).

The study of the classifying spaces of Kac-Moody groups began in Kitchloo's thesis [33] and continues with an eye toward connections to p-compact groups (c.f [2–4, 8, 34, 35]). There has also been an outgrowth (c.f. [9, 10, 13, 23, 25, 42]), more or less starting from [46], of geometric group theory that directly applies to discrete Kac-Moody groups. This paper is inspired by techniques from both bodies of literature and attempts to be friendly to all users. Methods tend toward the abstract and rely on functorial combinatorics, in the spirit of [40, 47]. We hope this presentation evokes a feeling of intimacy with the combinatorial group theory roots of geometric group theory and the combinatorial underpinnings of spaces modeled on CW-complexes.

1.1. **Summary of results.** Over the complex numbers, topological Kac-Moody groups are constructed by integrating, typically infinite dimensional, Kac-Moody Lie algebras [29]. Kac-Moody Lie algebras are defined via generators and relations encoded in a **generalized Cartan matrix** defined as a square integral matrix $A = (a_{ij})_{1\le i,j\le n}$ such that $a_{ii} = 2$, $a_{ij} \le 0$ and $a_{ij} = 0 \iff a_{ji} = 0$ [36]. Discrete Kac-Moody groups can be constructed in a manner analogous to Chevalley groups via a generalization of the Steinberg presentation. Tits [46] constructs a (minimal, split) Kac-Moody group functor, $K(R)$, from the category of commutative rings with unit to the category of groups that is the unique such functor when restricted to fields. Unless otherwise stated, when we say Kac-Moody group we mean some image of such a functor or the corresponding minimal topological group [30] whose underlying abstract group is $K(\mathbb{C})$ and A will denote the associated generalized Cartan matrix.

Kac-Moody groups have a standard maximal torus T of rank $2n - rank(A)$ common to standard parabolic subgroups indexed by subsets of $\{1, 2, \ldots n\}$. Each parabolic subgroup has a semi-direct product decomposition $P_I \cong G_I \ltimes U_I$. These G_I are called Levi component subgroups. If G_I is finite dimensional, then it is reductive of Lie type and we say I, G_I, U_I and P_I have finite type. The structure

of a BN-pair (see 2.2, 2.8) yields a homotopy decomposition for a Kac-Moody group K in terms of parabolic subgroups

$$\text{hocolim}_{I\in\mathbf{S}} BP_I \xrightarrow{\sim} BK \tag{1}$$

where $\mathbf{S}$ is the poset of subsets of $\{1, 2, \ldots n\}$ of finite type ordered by inclusion. Finite type U_I are known to be contractible in the complex topological case [36]. The semi-direct product decomposition and (1) imply that inclusion of the Levi component subgroups induces a homotopy decomposition for topological Kac-Moody group classifying spaces

$$\text{hocolim}_{I\in\mathbf{S}} BG_I \xrightarrow{\sim} BK \tag{2}$$

where G_I are reductive, Lie, Levi component subgroups, c.f. [33–35]. This decomposition has been one of the fundamental tools for the study BK using homotopy theory [3, 4, 8, 34].

In contrast, finite type $U_I(\mathbb{F}_{p^k})$ are far from being contractible (see 4.3), but we show that all U_I can be presented in terms finite dimensional unipotent groups.

Theorem 1. *Let $U^+ = U_\emptyset$ be the positive "unipotent" subgroup of a discrete Kac-Moody group whose Weyl group is viewed as a poset $\mathbf{W}$ with respect to the weak Bruhat order (10). Then, the canonical map induced by inclusion of subgroups*

$$\mathit{hocolim}_{\mathbf{W}} BU_w \xrightarrow{\sim} BU^+ \tag{3}$$

induces a homotopy equivalence where U_v (15) are finite dimensional unipotent algebraic subgroups. More generally, $\mathit{hocolim}_{\hat{U}_I^+\cdot\mathbf{W}} BU_{(\hat{u},w)} \xrightarrow{\sim} BU_I^+$ where $\hat{U}_I^+ \cdot \mathbf{W}$ is a poset with a contractible geometric realization and these $U_{(\hat{u},w)}$ are unipotent.

The proof relies on the combinatorics of the RGD system (see 2.3) associated to a Kac-Moody. This theoretical framework is natural w.r.t. the construction of Tits [46] and applies to many variants Kac-Moody groups as well (c.f [1, 10] for discussion). Note that (3) induces a known [1, 10] colimit presentation $\text{colim}_W U_w \cong U^+$ conjectured by Kac and Petersen [30]. However, the induced (see Remark 2 in 2.7) colimit presentations corresponding to $I \neq \emptyset$ are new. Theorem 1 will be shown by using alternative homotopy decompositions indexed over posets with finite chains given in Lemma 2.

A key technical point, which should be of independent interest, relates diagrams over the poset $\mathbf{W}$ with certain diagrams over the poset $\mathbf{W_S}$ (see 2.6) which underlies the Davis complex [12] of a Coxeter group.

Theorem 2. *For any Coxeter group W, the longest element functor $L : \mathbf{W_S} \to \mathbf{W}$ pullbacks homotopy colimits, i.e the natural map $\mathit{hocolim}_{\mathbf{W_S}} DL \longrightarrow \mathit{hocolim}_{\mathbf{W}} D$ is a weak homotopy equivalence* for any *diagram of spaces $D : \mathbf{W} \to \mathbf{Spaces}$.*

This immediately implies that the Davis complex is contractible since the poset $\mathbf{W}$ has the identity of W as its initial object. When $W \cong (\mathbb{Z}/2\mathbb{Z})^{*n}$, Theorem 2 lets us replace a diagram over an infinite depth tree ($\mathbf{W}$) with a canonical diagram over depth one tree ($\mathbf{W_S}$) obtained via barycentric subdivision. Theorem 2 allows this procedure to be extended to posets indexed by more general Coxeter groups.

Theorem 1 and the fact that unipotent algebraic groups over fields of characteristic p have trivial $\mathbb{F}_q$-homology lead to the following.

Theorem 3. *Let K be a topological (minimal, split) complex Kac-Moody group and $K(\overline{\mathbb{F}}_p)$ be the image of Tits's Kac-Moody group functor, $K(-)$, such that $K = K(\mathbb{C})$ (properly topologized). There exists a map $BK(\overline{\mathbb{F}}_p) \to BK$ that induces a $\mathbb{F}_q$-homology isomorphism for any prime $q \neq p$ so that localizing with respect to $\mathbb{F}_q$-homology gives a homotopy equivalence*

$$BK(\overline{\mathbb{F}}_p)^\wedge_q \xrightarrow{\sim} (BK^\wedge_q). \tag{4}$$

For Lie groups, this result was proved by Friedlander and Mislin [22]. For reductive Lie groups and φ^k the k-th power of the Frobenius map on $\overline{\mathbb{F}}_p$, the self-map $B(G(\varphi^k))^\wedge_q$ on $BG(\overline{\mathbb{F}}_p)^\wedge_q$ is the p^k-th unstable Adams operation, ϕ^k, on $BG^\wedge_q$, up to homotopy. Moreover, homotopy fixed points $(BG^\wedge_q)^{h\psi^k} \simeq BG(\overline{\mathbb{F}}_p)^{\wedge\, h\varphi^k}_q$ coincide with ordinary fixed points $BG(\mathbb{F}_{p^k})^\wedge_q \simeq (BG(\overline{\mathbb{F}}_p)^\wedge_q)^{\varphi^k}$ [21, 22].

Specifically, we can summarize this theory for the Lie case in the following theorem.

Theorem A ([21, 22])**.** *Let G be a complex reductive Lie group and $G(\mathbb{F}_{p^k})$ be of the same type. Then, there is a map*

$$BG(\overline{\mathbb{F}}_p) \xrightarrow{\overset{q}{\sim}} BG \tag{5}$$

inducing a $\mathbb{F}_q$-homology isomorphism for any prime $q \neq p$. The homotopy commutative diagram

$$\begin{array}{ccc} BG^\wedge_q & \xrightarrow{\psi^k} & BG^\wedge_q \\ \downarrow & & \downarrow \\ BG(\overline{\mathbb{F}}_p)^\wedge_q & \xrightarrow{BG(\varphi^k)} & BG(\overline{\mathbb{F}}_p)^\wedge_q \end{array} \tag{6}$$

where ψ^k is the p^k-th unstable Adams operation on the classifying space of G and $\varphi^k := (-)^{p^k}$ is the k^{th} Frobenius recovers a homotopy equivalence

$$BG(\mathbb{F}_{p^k})^\wedge_q \xrightarrow{\sim} (BG^\wedge_q)^{h\psi^k} \ . \tag{7}$$

This work began to investigate the possibility of extending the fact that $(BG^\wedge_q)^{h\psi^k} \simeq BG(\mathbb{F}_{p^k})^\wedge_q$ to Kac-Moody groups. In order to even approach this question, we need a notion of p^k-th unstable Adams operation. In [4], such maps are constructed for rank 2 Kac-Moody groups. We use Theorem 3 to construct them for arbitrary Kac-Moody groups.

Theorem 4. *Let K be a topological Kac-Moody group over $\mathbb{C}$ with standard Levi components G_I and Weyl group W. Then there are p^k-th q-local unstable Adams operations $\psi^k : BK^\wedge_q \to BK^\wedge_q$ for any prime $q \neq p$ compatible with the p^k-th unstable Adams operations, ψ^k_I, on the finite type Levi component classifying spaces. If there is no element of order p in W, these local unstable Adams operations can be assembled into a global unstable Adams operation $\psi^k : BK \to BK$.*

This definition of lets us consider $(BK^\wedge_q)^{h\psi^k}$ for K a Kac-Moody group. Recent work by Kishimoto and Kono [32] facilitates the determination the induced maps on cohomology for the Levi component homotopy fixed point spaces. By computing in rank 2, we provide many examples when $(BK^\wedge_q)^{h\psi^k}$ and $BK(\mathbb{F}_{p^k})^\wedge_q$ have distinct homotopy types in sharp contrast to the Lie case.

Theorem 5. *For K a rank 2, infinite dimensional complex Kac-Moody group and q an odd prime distinct from p, $H^*((BK^\wedge_q)^{h\psi^k}, \mathbb{F}_q) = H^*(BK(\mathbb{F}_{p^k})^\wedge_q, \mathbb{F}_q)$ if and only if they both vanish.*

Remark 1. *The distinction between $(BK^\wedge_q)^{h\psi^k}$ and $BK(\mathbb{F}_{p^k})^\wedge_q$ is noted by Aguadé and Ruiz in* [5] *where rank two Kac-Moody groups are studied. Their work occurred independently and uses algebraic group cohomology calculations for fully compute*

$H^*(BK(\mathbb{F}_{p^k}), \mathbb{F}_q)$ in many cases. Here Theorem 5 is observed through cohomology calculations with compact Lie group classifying spaces and non-trivial calculations of $H^*(BK(\mathbb{F}_{p^k}), \mathbb{F}_q)$ are not attempted.

1.2. **Organization of the paper.** Section 2 overviews background material from geometric group theory and methods to manipulate homotopy colimits diagrammatically we need in later arguments. This leads to a proof of Theorem 2 (3.1) and a reduction of Theorem 1 to the more naturally proved Lemmas 2 and 3 (3.2). We return to the Kac-Moody case in 3.3 which details applications to Kac-Moody groups over a fields of characteristic p obtained upon localization at a prime $q \neq p$, including Theorem 3. These results are employed in 3.4 to define unstable Adams operations for general Kac-Moody groups. Section 4 proves Theorem 5 and outlines explicit calculations of the group cohomology of $U^+(\mathbb{F}_{p^k}) \leq K(\mathbb{F}_{p^k})$ at p in specific cases where the Weyl group is a free product.

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2. Combinatorial tool kit

Like Lie groups, Kac-Moody groups have (typically infinite) Weyl groups that underlie combinatorial structures on subgroups including BN-pairs and root group data. In 2.1-2.3, the aspects of these structures we will employ are assembled. We will also collect tools for manipulating homotopy colimits (2.4-2.6) which relate to colimit presentations of groups via Seifert-van Kampen theory (2.7). With a homotopy theoretic framework in place, 2.8 quickly reviews two known homotopy decompositions associated to Coxeter groups and BN-pairs.

2.1. **Coxeter groups.** Because of their role in BN-pairs and RGD systems, the structure of Coxeter groups will be important to the arguments here. For our purposes, a Coxeter group will be a *finitely* generated group with a presentation

$$\langle s_1, s_2, \dots s_n | s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1 \rangle \tag{8}$$

for $1 \leq i, j \leq n$ and fixed $2 \leq m_{ij} = m_{ji} \leq \infty$ with $m_{ij} = \infty$ specifying a vacuous relation. For example, the presentation the Weyl group of a Kac-Moody group is determined by its generalized Cartan matrix $A = (a_{ij})_{1 \leq i,j \leq n}$, i.e.

$$m_{ij} = \begin{cases} 2 & a_{ji}a_{ij} = 0 \\ 3 & a_{ji}a_{ij} = 1 \\ 4 & a_{ji}a_{ij} = 2 \\ 6 & a_{ji}a_{ij} = 3 \\ \infty & a_{ji}a_{ij} \geq 4 \end{cases} \tag{9}$$

for all $i \neq j$. The solution to the word problem for Coxeter groups gives us a detailed picture of W.

Theorem B ([1])**.** *Let W be a Coxter group. For any words $\vec{v}$ and $\vec{w}$ in letters $s_1, s_2, \dots s_n$*

- *A word $\vec{w}$ is reduced in W if and only if it cannot be shortened by a finite sequence of s_is_i deletions and replacing of the length $m_{ij} < \infty$ alternating words $s_is_j \dots$ by $s_js_i \dots$ or vice versa.*
- *Reduced words $\vec{v}$ and $\vec{w}$ are equal in W if and only if $\vec{v}$ transforms into $\vec{w}$ via a finite sequence of length $m_{ij} < \infty$ alternating word by replacements as describe above.*

Thus, every element of a Coxeter group has a well-defined reduced word length. This provides a poset structure on W, called the **weak Bruhat order**, defined as

$$v \leq w \iff w = vx \text{ and } l(w) = l(v) + l(x) \tag{10}$$

where $x, v, w \in W$ and $l(y)$ is the reduced word length of y. This poset will be important to us because it dictates the intersection pattern of the subgroups U_w (15) of a group with RGD (see 2.3).

For any $I \subseteq S$ define $W_I := \langle \{s_I\}_{i \in I \subseteq S} \rangle$. For all I in the poset

$$\mathbf{S} := \{I \subseteq S \| W_I| < \infty\} \tag{11}$$

ordered by inclusion wW_I has a unique longest word w_I. We call $I \in \mathbf{S}$ and their associated W_I **finite type**. This notion coincides with with our definition on finite type subsets $J \subseteq \{1, 2, \dots n\}$ associated to a Kac-Moody group. Specifically, the submatrix $A_J := (a_{ij})_{i,j \in J}$ of A determines a Weyl group W_J group for P_J and G_J via (9) and G_J is of Lie type exactly when $|W_J| < \infty$ [36]. We emphasize in particular that $K = G_{\{1,2,\dots n\}}$ is of Lie type if and only if its Weyl group given by (9) is finite.

2.2. BN-pairs.

Here we follow *Buildings* [1].

Definition 1. *A group G is a BN-pair if it has data (G, B, N, S) where B and N are distinguished normal subgroups that together generate G. Furthermore, $T := B \cap N \trianglelefteq N$ and $W := N/T$ is generated by S so that for $\overline{s} \in s \in S$ and $\overline{w} \in w \in W$:*
BN1: $\overline{s}B\overline{w} \subset B\overline{sw}B \cup B\overline{w}B$
BN2: $\overline{s}B\overline{s^{-1}} \not\subseteq B$.

Note that the above sets are well-defined as $B \geq T \trianglelefteq N$. It is common in the literature, and will be common in this paper, to drop the bars and avoid reference to particular representatives of elements of W. For Kac-Moody groups, T is the maximal torus, N is the normalizer of T, $|S|$ is the size of the generalized Cartan matrix and B is the standard Borel subgroup defined analogously to the Lie case. In line with 2.1, we will further require that S is a finite set.

Important properties of BN-pairs include that W is a Coxeter group and G admits a **Bruhat decomposition**

$$G = \coprod_{w \in W} BwB$$

so that all subgroups of G containing B are of the form $P_I = \coprod_{w \in W_I} BwB$ where $I \subseteq S$ and W_I is generated by I. These subgroups are called **standard parabolic subgroups** and inherit the the structure of a BN-pair with data $(P_I, B, N \cap P_I, I)$ and Weyl group W_I. The BN-pair axioms then lead [36] to **generalized Bruhat decompositions**

$$G = \coprod_{w \in W_J \backslash W / W_I} P_J w P_I. \tag{12}$$

2.3. **Groups with root group data.** Much of the work in this paper could be adapted to refined Tits systems [30] but we prefer to work with root group data (RGD) systems. Notably the framework of RGD systems has been used to prove [1, 10] the colimit presentation induced by Theorem 1 (see Remark 2 in 2.7) which was conjectured by Kac and Petersen in [30].

Briefly, a RGD system for a group G is a given by a tuple $(G, \{U_\alpha\}_{\alpha\in\Phi}, T)$ for Φ associated to a Coxeter system (W, S). The elements of the set Φ are called roots and U_α are nontrivial subgroups of G known as **root subgroups**. The RGD subgroups generate G, i.e. $G = \langle T, \{U_\alpha\}_{\alpha\in\Phi}\rangle$. In the case of Tits's Kac-Moody group functor, U_α are isomorphic to the base ring as a group under addition and T is the standard maximal torus isomorphic to a finite direct product of the multiplicative group of units of the base ring. As the complete definition is somewhat involved and the arguments here can be followed with the Kac-Moody example in mind, we refer the reader to [1] for the standard definition of a RGD system and further details on the contents of this subsection but note that [10] provides an alternative formulation.

The RGD structure regulates the conjugation action of W on U_α. In particular, Φ has a W-action and is the union of the orbits of the simple roots, α_i, corresponding to the elements, s_i, of S. For $n \in w \in W$

$$U_{w\alpha} = nU_\alpha n^{-1}.$$

In the Kac-Moody case, $\Phi = W\{\alpha_i\}_{1\le i\le n} \subseteq \sum_{1\le i\le n} \mathbb{Z}\alpha_i$ with the W-action given by

$$s_j(\sum_{i=1}^{k} n_i\alpha_i) = \sum_{1\le i\ne j\le k} n_i\alpha_i - (n_j + \sum_{1\le i\ne j\le k} n_i a_{ij})\alpha_j. \tag{13}$$

where a_{ij} are entries in the generalized Cartan matrix.

The set of roots Φ is divided into positive and negative roots so that $\Phi = \Phi^- \coprod \Phi^+$ which in the Kac-Moody case corresponds to the subsets of Φ with all negative or all positive coefficients. Define

$$U^\pm = \langle U_\alpha; \alpha \in \Phi^\pm\rangle. \tag{14}$$

The group G carries the structure of a BN-pair for either choice of $B = TU^\pm$ and N the normalizer of T. There are also well-defined

$$U_w = \langle U_{\alpha_{i_1}}, U_{s_{i_1}\alpha_{i_2}}, \ldots U_{s_{i_1}\ldots s_{i_{k-1}}\alpha_{i_k}}\rangle \tag{15}$$

where $\Theta_w := \{\alpha_{i_1}, \alpha_{i_2}, \ldots, ws_{i_l}\alpha_{i_l}\} = \Phi^+ \cap w\Phi^-$ is well-defined for any reduced expression for $w = s_{i_1}\ldots s_{i_k}$. Moreover, the multiplication map

$$U_{\alpha_{i_1}} \times U_{s_{i_1}\alpha_{i_2}} \times \ldots \times U_{ws_{i_l}\alpha_{i_l}} \xrightarrow{m} U_w \tag{16}$$

is an isomorphism of sets for any choice of reduced expression for $w = s_{i_1}\ldots s_{i_k}$. For any index set A

$$\bigcap_A U_w = U_{\inf_{\mathbf{W}}\{A\}} \tag{17}$$

where the greatest lower bound is taken with respect to the weak Bruhat order (10).

For a group with RGD, there is a symmetry between the positive and negative roots, as in the Kac-Moody case. In particular, there is another RGD system for G, $(G, \{U_\alpha\}_{\alpha\in-\Phi}, T)$, with the positive and negative root groups interchanged. This

induces a twin BN-pair structure on G which guarantees a **Birkhoff decomposition**

$$G = \coprod_{w \in W} B^+ w B^-$$

which when combined with the Bruhat decomposition for P_I (12) and the fact that $B = U^+ T$ easily leads to a **generalized Birkhoff decomposition**

$$G = \coprod_{wW_J \in W/W_J} U^+ w P_J^- \tag{18}$$

where $U^+ w P_J^- = \coprod_{v \in wW_J} U^+ w B^-$ (see [20] for details). Using the standard RGD axioms, this symmetry between positive and negative is somewhat subtle. Known proofs employ covering space theory [1, 10].

The loc. cit. covering homeomorphisms imply for $n \in w \in W$

$$U^\pm \cap n U^\mp n^{-1} = U^\pm \cap w B^\mp w^{-1} = U_w^\pm \tag{19}$$

where $U_w^+ := U_w$ and $U_w^- = \langle U_{-\alpha_{i_1}}, U_{-s_{i_1}\alpha_{i_2}}, \dots U_{-s_{i_1} \dots s_{i_{k-1}} \alpha_{i_k}} \rangle$. More generally (see Lemma 1), for all $J \in \mathbf{S}$ (11)

$$U^\pm \cap w P_J^\mp w^{-1} = U_{w_J}^\pm \tag{20}$$

where w_J is the longest word in wW_J.

The groups U_w also provide improved Bruhat decompositions because

$$B^\pm w B^\pm = U_w^\pm w B^\pm$$

where expression on the left hand side factors *uniquely* for a fixed choice of $w \in nT \in W$, c.f. [1, Lemma 8.52], i.e.

$$G = \coprod_{w \in W} U_w^+ w B^+ = \coprod_{w \in W} U_w^- w B^- \tag{21}$$

so that all $g \in G$ factor uniquely as $g = uwb$ where $u \in U_w^\pm$, $b \in B^\pm$ and w is an element in any fixed choice of coset representatives. Parabolic subgroups inherit similar expressions.

As previously mentioned, all groups with RGD have the structure of a twin BN-pair. This structure can be used to define Levi component subgroups

$$G_I = P_I^- \cap P_I^+. \tag{22}$$

The G_I inherit a root group data structure $(G_I, \{U_\alpha\}_{\alpha \in \Phi_I}, T)$ where $\Phi_I := \{\alpha \in \Phi | U_\alpha \cap G_I \neq \{e\}\}$.
When I has finite type [10], this leads to the semi-direct product decomposition

$$P_I^\pm \cong G_I \ltimes U_I^\pm \tag{23}$$

known as the Levi decomposition with

$$G_I = \langle T, \hat{U}_I^+, \hat{U}_I^+ \rangle$$

for $\hat{U}_I^\pm = U^\pm \cap G_I$. Most of our principal applications will only require (23) for finite type I. For Kac-Moody groups, (23) holds for all I. See [10] for further discussion of when this decomposition is known to exist for general RGD systems.

We close the subsection with a simple but important lemma used in the proof of Theorem 1.

Lemma 1. *For P_J^- of finite type*

$$U_{wP_J^-} = U_{w_J} \tag{24}$$

where $U_{wP_J^-}$ is the stabilizer of $\{wP_J^-\}$ under the left multiplication action of U^+ on G/P_J^-, w_J is the longest word in wW_J, and U_w is defined in (15).

Proof: Note that $uwP_J^- = wP_J^-$ and $u \in U^+$ if and only if $u \in wP_J^- w^{-1} \cap U^+$. Recall (21) the (improved) Bruhat decomposition for P_J^-

$$P_J^- = \coprod_{v \in W_J} U_v^- vB^-. \tag{25}$$

We compute

$$\begin{aligned} wP_J^- w^{-1} = w_J P_J^- {w_J}^{-1} &= \bigcup_{v \in W_J} w_J U_v^- vB^- {w_J}^{-1} \\ &= \bigcup_{v \in W_J} (w_J U_v^- {w_J}^{-1}) w_J vB^- {w_J}^{-1}. \end{aligned} \tag{26}$$

As $(w_J U_v^- {w_J}^{-1}) \subseteq U^+$ for all $v \in W_J$ we have

$$w_J U_v^- vB^- {w_J}^{-1} \cap U^+ \cong w_J vB^- {w_J}^{-1} \cap U^+.$$

Each $w_J vB^- {w_J}^{-1} \cap U^+$ isomorphic as a set to

$$w_J vB^- \cap U^+ w_J \subseteq U^+ w_J vB^- \cap U^+ w_J B^- \tag{27}$$

via right multiplication by w_J. Together (25) and (27) imply

$$w_J vB^- {w_J}^{-1} \cap U^+ \neq \emptyset \Rightarrow v = e. \tag{28}$$

Combining (26-28) we have

$$\begin{aligned} U_{wP_J^-} = wP_J^- w^{-1} \cap U^+ &= \bigcup_{v \in W_J} (w_J U_v^- vB^- {w_J}^{-1} \cap U^+) \\ &= w_J U_e^- eB^- {w_J}^{-1} \cap U^+ \\ &= w_J B^- {w_J}^{-1} \cap U^+ = U_{w_J B^-}. \end{aligned}$$

Now, $(w_J B^- {w_J}^{-1} \cap U^+) = U_{w_J B^-}$ is known to be U_{w_J} (19) via covering arguments that appear independently in [1] and [10]. $\square$

2.4. **Pulling back homotopy colimits.** If $F : \mathbf{J} \to \mathbf{I}$ is a fixed functor, then we say F **pulls back homotopy colimits** if *for any* diagram of spaces $D : \mathbf{I} \to$ **Spaces** the natural map

$$\mathrm{hocolim}_{\mathbf{J}} DF \xrightarrow{\mathrm{hocolim}_F} \mathrm{hocolim}_{\mathbf{I}} D$$

is a weak homotopy equivalence. We will use Theorem 6 (see Section 2.5) to pull-back homotopy colimits over appropriate functors by assembling pullbacks over subcategories. For such a functor and any $i \in \mathbf{I}$ define the category $i \downarrow F$ as having objects

$$Objects(i \downarrow F) = \{(i \to i', j') | F(j') = i', i \to i' \in Hom_{\mathbf{I}}\}$$

and morphisms

$$\begin{aligned} & Hom_{i \downarrow F}((i \xrightarrow{g} i'_1, j'_1), (i \xrightarrow{f} i'_2, j_2)) \\ = \; & \{(i'_1 \xrightarrow{F(h)} i'_2, j'_1 \xrightarrow{h} j'_2 | F(h)g = f \in Hom_{\mathbf{I}}, h \in Hom_{\mathbf{J}})\}, \end{aligned}$$

i.e. morphisms are pairs of vertical arrow that fit into the following diagram

$$\begin{tikzcd}
& i'_1 = F(j'_1) \arrow[dd, "F(h)"'] & & j'_1 \arrow[ll, maps to] \arrow[dd, "h"] \\
i \arrow[ur, "g"] \arrow[dr, "f"'] & {} & & {} \arrow[ll, Leftarrow, maps to, "F"'] \\
& i'_2 = F(j'_2) & & j'_2 \arrow[ll, maps to]
\end{tikzcd} .$$

When F is the identity on $\mathbf{I}$, we use the notation $i \downarrow \mathbf{I}$ and this definition reduces to that of an under category.

Theorem C ([17, 28])**.** *Let $D : \boldsymbol{I} \to \mathbf{Spaces}$ be a diagram of spaces and $F : \mathbf{J} \to \boldsymbol{I}$ be a fixed functor, then the following are equivalent:*

- *for all D the canonical $hocolim_{\mathbf{J}} DF \xrightarrow{\sim} hocolim_{\boldsymbol{I}} D$ is a weak equivalence,*
- *for all $i \in \boldsymbol{I}$ the geometric realization of $i \downarrow F$ is weakly contractible.*

In the applications of this paper, $\mathbf{J}$ and $\mathbf{I}$ will be posets so that $i \downarrow F = F^{-1}(i \downarrow \mathbf{I})$.

2.5. **Homotopy colimits over subdiagrams.** At various times in this paper, we will wish to represent homotopy colimit presentations of a space in terms of homotopy colimits over subdiagrams of the main diagram. This subsection will present tools to do this.

An explicit model for the homotopy colimit of a small diagram $D : \mathbf{C} \to \mathbf{Top}$ of (topological) spaces $\mathrm{hocolim}_{\mathbf{C}} D$ may be given as the union of

$$X_n = \frac{\coprod_{c \to c_1 \ldots \to c_{k \leq n} \in \mathbf{C}} F(c) \times \Delta^k}{\sim} \tag{29}$$

where Δ^k is the k-simplex and the relations correspond to the composition in $\mathbf{C}$ [7]. We call this the **standard model** of a homotopy colimit. Define the **geometric realization** or classifying space of a category as $|\mathbf{C}| \simeq \mathrm{hocolim}_{\mathbf{C}}\{*\}$ for $\{*\}$ the one point space.

We will also define a cover of a category $\mathbf{C}$ by subcategories $\{\mathbf{C}_i\}_{i \in I}$.

Definition 2. *A cover of a small category $\boldsymbol{C}$ is a collection of subcategories $\{\boldsymbol{C}_i\}_{i \in I}$ such that, taking standard models (29), $\{|\boldsymbol{C}_i|\}_{i \in I}$ covers $|\boldsymbol{C}|$ (2.5).*

We may also choose to enrich a cover by giving I the structure of a poset such that $i \mapsto \mathbf{C}_i$ is a (commutative) diagram of inclusions of small categories. Such a cover is given as a functor $\mathcal{U} : \mathbf{I} \to \mathbf{SubCat}(\mathbf{C})$ from a poset into the category subcategories of $\mathbf{C}$. If there are no repetitions of $\mathbf{C}_i$'s, then I may be canonically given the poset structure induced by inclusion of subcategories.

We will be interested in the taking homotopy colimits over subdiagrams

$$D_i = D|_{\mathbf{C}_i} : \mathbf{C}_i \longrightarrow \mathbf{Top}$$

and one of our main computational tools is provided by the following theorem.

Theorem 6. *Let $D : \boldsymbol{C} \to \mathbf{Top}$ be a diagram of topological spaces and $\mathcal{U} : \boldsymbol{I} \to$ $\mathbf{SubCat}(\mathbf{C})$ be a cover $\{\boldsymbol{C}_i\}_{i \in \boldsymbol{I}}$ of $\boldsymbol{C}$ such that*

- *$\boldsymbol{I}$ has all greatest lower bounds and*
- *$\mathcal{U}(\inf_A\{\boldsymbol{C}_a\}) = \cap_A \mathcal{U}(\boldsymbol{C}_a)$*

for all index sets A. Then, $hocolim_{\boldsymbol{I}}(hocolim_{\boldsymbol{C}_i} D_i)$ and $hocolim_{\boldsymbol{C}} D$ are canonically, weakly equivalent.

By Thomason's Theorem [44, Thereom 1.2], Theorem 6 may be reduced to the following.

Proposition 1. *Let $D : \boldsymbol{C} \to$ **Spaces** be a diagram of spaces (e.g. topological spaces, CW-complexes, or simplicial sets) and $\mathcal{U} : \boldsymbol{I} \to$ **SubCat(C)** be a cover $\{\boldsymbol{C}_i\}_{i\in \boldsymbol{I}}$ of $\boldsymbol{C}$ such that*

- *$\boldsymbol{I}$ has all greatest lower bounds and*
- *$\mathcal{U}(\inf_A\{\boldsymbol{C}_a\}) = \cap_A \mathcal{U}(\boldsymbol{C}_a)$*

for all index sets A. Then, the canonical projection of the Grothendieck construction (31) of $\mathcal{U}$ onto $\boldsymbol{C}$ pulls back homotopy colimits.

Proof: Recall that the Grothendieck construction of $\mathcal{U}$, which we will denote as $\mathbf{I} \ltimes \mathcal{U}$, is given by

$$\begin{aligned} Objects(\mathbf{I} \ltimes \mathcal{U}) &= \{(i,c)|i \in Objects(\mathbf{I}),\ c \in Objects(\mathbf{C}_i)\}, \\ Hom_{\mathbf{I}\ltimes\mathcal{U}}((i,c),(i',c')) &= \{(f,g)|f \in Hom_{\mathbf{I}}(i,i'),\ g \in Hom_{C_{i'}}(c,c')\} \end{aligned} \tag{30}$$

with composition given by pushing forward along $\mathcal{U}$, i.e. $(f_2,g_2)\circ(f_1,g_1) = (f_2f_1, g_2\mathcal{U}(g_1))$. The projection $P : \mathbf{I} \ltimes \mathcal{U} \to \mathbf{C}$ maps (i,c) to c. By Theorem C, it is sufficient to show that $|c \downarrow P| \simeq \{*\}$. Let $i(c)$ be the greatest lower bound for all categories in the cover containing c, then $c \downarrow P$ has initial object $(c \xrightarrow{id} c, (i(c), c))$. □

Dugger and Isaksen [16] present a model, up to weak equivalence, of a general topological space X in terms of an arbitrary open cover $\mathbf{U} = \{U_i\}_{i\in\mathbf{I}}$ and *finite* intersections of these U_i. This allows a version of Theorem 6 to verified depending only on finite greatest lower bounds, but this version will not be needed here and is less natural in the sense that it relies on the standard weak equivalences generating the model category structure.

Now, Theorem 6 allows the second criterion of Theorem C to be verified locally.

Proposition 2. *Let $F : \boldsymbol{J} \to \boldsymbol{I}$ be a map of posets such that $\boldsymbol{I}$ has all greatest lower bounds. If F restricted to the pullback of all closed intervals pull backs homotopy colimits, then F pull backs homotopy colimits.*

Proof: A closed interval $[i_1, i_2]$ is defined to be the full subcategory of $\mathbf{I}$ with object set $\{i|i_1 \to i \to i_2 \in \mathbf{I}\}$. The category $i \downarrow \mathbf{I}$ is covered by $\{[i,i']\}_{\exists i\to i'}$ and for any finite set of objects A the intersection $\cap_{i'\in A}[i,i'] = [i, \inf_{\mathbf{I}} A]$. Since covers and intersections pullback over functors, $\{F^{-1}[i,i']\}_{\exists i\to i'}$ covers $\mathbf{J}$ and is closed under intersection. By hypothesis, $|F^{-1}[i,i']| \simeq \{*\}$. Theorem 6 implies $|i \downarrow F = F^{-1}(i \downarrow \mathbf{I})|$ and $|i \downarrow \mathbf{I}|$ have the same weak homotopy type. The result follows as $i \downarrow \mathbf{I}$ has an initial object, namely the identity on i. □

2.6. **Transport categories.** Transport categories are a specialization of the Grothendieck construction [44] for a small diagram of small categories.

Definition 3. *Given a (small) diagram of sets $X : \boldsymbol{C} \to$ **Sets**, the transport category of X, denoted $\boldsymbol{Tr}(X)$ has*

$$\begin{aligned} Ob(\boldsymbol{Tr}(X)) &= \{(c, x_c)|c \in \boldsymbol{C}, x_c \in X(c)\} \\ Hom_{\boldsymbol{Tr}(X)}((c,x_c),(c',y_{c'})) &= \{f \in Hom_{\boldsymbol{C}}(c,c')|X(f)(x_c) = y_{c'}\} \end{aligned}$$

with composition induced by $\boldsymbol{C}$.

For any Coxeter group W with $\mathbf{S}$ the set of subsets of generators that generate finite groups (11), our fundamental example of a transport category is $\mathbf{Tr}(X)$ for $X : \mathbf{S} \to$ **Sets** defined via $I \mapsto W/W_I$. We call this category $\mathbf{W_S}$. Explicitly, $\mathbf{W_S}$ is a poset whose elements are finite type cosets and there exists a (unique) morphism from wW_I to vW_J precisely when $wW_I \subseteq vW_J$. Its geometric realization $|\mathbf{W_S}|$ is the Davis complex [12].

We also wish to record a proposition which is a specialization and enrichment of Thomason's Theorem [44, Thereom 1.2]. We indicate an explicit proof.

Proposition 3. *For any diagram of spaces over a transport category $D : \boldsymbol{Tr}(X) \to$ **Spaces**, there is a diagram of spaces $D' : \boldsymbol{C} \to$ **Spaces** over the underlying category such that $hocolim_{\boldsymbol{Tr}(X)} D$ and $hocolim_{\boldsymbol{C}} D'$ are canonically weakly equivalent.*

Proof: Let $D' : \mathbf{C} \to \mathbf{Spaces}$ be defined by $D'(c) := \mathrm{hocolim}_{X(c)} D(c, x_c)$. We have a natural weak equivalence

$$\mathrm{hocolim}_{\mathbf{Tr}(X)} D' \xrightarrow{\sim} \mathrm{hocolim}_{\mathbf{C}}(\mathrm{hocolim}_{X(c)} D|_{X(c)}) = \mathrm{hocolim}_{\mathbf{C}} D'$$

which can be observed by inspecting the standard models. More explicitly, if one takes standard models the map (31) is specified by the universal property of the homotopy colimit and the maps

$$D(c, x_c) \xrightarrow{D} D(c', y_{c'}) \longrightarrow \coprod_{z_{c'} \in X(c')} D(c', z_{c'})$$

where the second map is the obvious inclusion. □

Note that Proposition 3 gives a canonical equivalence between $\mathrm{hocolim}_{\mathbf{C}} X$ and $|\mathbf{Tr}(X)|$.

2.7. **Coset geometry and colimits of groups.** Here we will see how homotopy colimit calculations in terms of subdiagrams apply to honest colimits.

Theorem D ([14], Corollary 5.1)**.** *Let $D : \boldsymbol{I} \to \mathbf{Spaces}_*$ be a diagram of pointed connected spaces such that $\boldsymbol{I}$ has an initial object. Then, there is a natural isomorphism $\pi_1 hocolim_{\boldsymbol{I}} D \cong colim_{\boldsymbol{I}} \pi_1 D$.*

When $D : \mathbf{I} \to$ **Groups** is diagram of inclusions of subgroups of H, we refer to the standard model of the homotopy fibre of $\mathrm{hocolim}_{\mathbf{I}} BD \to BH$ given by $\mathrm{hocolim}_{\mathbf{I}} H/D(i)$ as the **coset geometry** of the cocone. We call the transport category (see 2.6) for the functor $i \mapsto H/D(i)$ the **poset form of the coset geometry** since its geometric realization is canonically equivalent to the coset geometry. This represents a mild generalization of a notion of coset geometry employed by others, such as Tits [45] and Caprace and Remy [10]. For instance, our notion permits diagrams whose image in the lattice of subgroups is not full and our poset is directed in the oppositive direction.

We can always add the trivial group to any diagram of inclusions of groups in an initial position without affecting the colimit of that diagram. Thus, verifying any presentation of a group as a colimit of subgroups reduces to homotopy knowledge of the coset geometry. An example of this technique gives the following additional corollary.

Corollary 1. *Let $D : \boldsymbol{I} \to$ **Groups** be a diagram of the subgroups of H with each map an inclusion of subgroups such that $|\boldsymbol{I}|$ is simply connected. The canonical $colim_{\boldsymbol{I}} D \xrightarrow{\sim} H$ is an isomorphism if and only if its coset geometry is simply connected.*

Proof: We may artificially add an initial object, $\bullet$, to $\mathbf{I}$ and call this new category $\mathbf{I} \cup \bullet$. We can extend D to $\mathbf{I} \cup \bullet$ by sending $\bullet$ to the trivial group. Up to homotopy, the space $Y := \mathrm{hocolim}_{\mathbf{I} \cup \bullet} BD$ is obtained from $X := \mathrm{hocolim}_{\mathbf{I}} BD$ by coning off the subspace $Z := \mathrm{hocolim}_{\mathbf{I}} \{*\}$, the homotopy colimit base points. Using Theorem D, the fundamental group $\pi_1(\mathrm{hocolim}_{\mathbf{I} \cup \bullet} BD) \cong \mathrm{colim}_{\mathbf{I}} D$. Thus, $\mathrm{colim}_{\mathbf{I}}$ agrees with $\pi_1(\mathrm{hocolim}_{\mathbf{I}} BD)$ if and only if in the pushout $Y \simeq X \bigcup_Z CZ$ has $\pi_1(Z) = 0$. □

Compare [11, Lemma 1.3.1] and [26, §3]. For example, as observed in [10], if W is a Coxeter group with $\mathbf{S}_2$ the set of subsets of generators of cardinality at most 2 that generate finite groups, then the colimit presentation $\mathrm{colim}_{\mathbf{S}_2} W_I \cong W$, which is immediate from the presentation of W (8), implies that the associated coset geometry is simply connected.

Remark 2. *When the conditions for Corollary 1 are satisfied, a homotopy decomposition of a classifying space in terms of the classifying spaces of subgroups induces a colimit presentation. Though not stated explicitly, the homotopy decompositions of Theorem 1, Lemma 2, and Corollary 4 as well as the known homotopy decompositions (31-32) all induce colimit presentations. Of course, a homotopy decomposition of a classifying space in terms of the classifying spaces of subgroups is a strictly stronger condition characterized by having a contractible coset geometry whereas a colimit presentation only requires a simply connected coset geometry. For instance, $colim_{\mathbf{S}_2} W_I \cong W$ is induced by a homotopy decomposition of BW only if $\mathbf{S}_2 = \mathbf{S}$.*

2.8. **Known homotopy decompositions.** For any Coxeter group W with $\mathbf{S}$ (11) the poset of subsets of generators that generate finite groups

$$\text{hocolim}_{I \in \mathbf{S}} BW_I \xrightarrow{\sim} BW \tag{31}$$

is a homotopy equivalence including, trivially, the cases where W is finite. For non-trivial decompositions, the associated coset geometry is Davis's version of the Coxeter complex, $|\mathbf{W_S}|$, which is known to be contractible (c.f. [39]). Theorem 2 provides a combinatorial proof. Likewise, for any discrete BN-pair the canonical map induced by inclusion of subgroups

$$\text{hocolim}_{I \in \mathbf{S}} BP_I \xrightarrow{\sim} BG \tag{32}$$

is a homotopy equivalence including, trivially, the cases where W is finite. Here non-trivial decompositions have Davis's version of the Tits building as coset geometries. For completeness, we also note that (31) and (32) can be deduced from the contractibility of the usual Coxeter complex (resp. Tits building) inductively because this provides a method to verify (32) for complex topological Kac-Moody groups. More specifically, for any topological G, a BN-pair with closed parabolic subgroups and infinite Weyl group W the canonical map $\text{hocolim}_{I \in \mathbf{R}} BP_I \longrightarrow BG$ for $\mathbf{R}$ the set of all proper subsets of the (finite) generating set of W has contractible coset geometry [41]. Thus, this map is a weak homotopy equivalence. Because parabolic subgroups inherit the structure of a BN-pair and inductively (on $|I|$) have homotopy decompositions (32) in terms of finite type parabolics, Theorem 6 may be applied to obtain (32) for BG whenever all parabolic subgroups are closed. Decompositions such for topological BN-pairs are implicit in [33–35].

We close noting that (31-32) are sharp in the sense that any proper subposet of $\mathbf{S}$ will not suffice to give a homotopy decomposition.

3. Main Results

3.1. **The proof of Theorem 2.** Our proof of Theorem 2 will inductively pullback homotopy colimits over all closed intervals and apply Proposition 2. For instance, the following corollary which will be used in 4.3 is immediate from our proof by Proposition 2.

Corollary 2. *With the definitions of Theorem 2, let $\mathbf{X} \subseteq \mathbf{W}$ be a full subposet covered by a collection of intervals $[v, w]$, then the longest element map $L|_{L^{-1}\mathbf{X}}$ pulls back homotopy colimits.*

We may also extend Theorem 2 to situations where $\mathbf{W}$ is the fundamental domain of a group action on a poset. We are especially interested in $V \cdot \mathbf{W}$ which for any $V \leq U^+$ is defined to be the V-orbit under left multiplication of $\mathbf{W}$ realized as the poset of subgroups $\{U_w\}_{w \in W}$ ordered by inclusion within the poset of cosets of subgroups of U^+. Also define $V \cdot \mathbf{W_S}$ so that L extends to a V-equivariant functor $V \cdot L : V \cdot \mathbf{W_S} \to V \cdot \mathbf{W}$, i.e. wW_I corresponds to U_{w_I}.

Corollary 3. *Let* $\mathbf{W}$ *be realized as the poset of subgroups* $\{U_w\}_{w\in W}$ *ordered by inclusion. Then* $V\cdot L : V\cdot \mathbf{W_S} \to V\cdot \mathbf{W}$ *pulls back homotopy colimits for any* $V \le U^+$.

Proof: Observe that $V\cdot \mathbf{W}$ is the transport category for $X : \mathbf{W} \to \mathbf{Sets}$ defined by $w \mapsto V/(V\cap U_w)$ and $V\cdot \mathbf{W_S}$ is that transport category for $X_\mathbf{S} : \mathbf{W_S} \to \mathbf{Sets}$ defined by $wW_I \mapsto V/(V\cap U_{w_I})$. As in the proof of Proposition 3, any fixed functor $V\cdot D : V\cdot \mathbf{W} \to \mathbf{Spaces}$ is associated to $D : \mathbf{W} \to \mathbf{Spaces}$ defined by $D(w) := \coprod_{X(w)} D(v(V\cap U_w))$. If $V\cdot D$ is pulled back along $V\cdot L$, then we obtain $(V\cdot D)(V\cdot L) : V\cdot \mathbf{W_S} \to \mathbf{Spaces}$ which is associated to $DL : \mathbf{W_S} \to \mathbf{Spaces}$ by the same procedure. Thus, we have a diagram

$$\begin{tikzcd} \mathrm{hocolim}_{V\cdot \mathbf{W_S}}(V\cdot D)(V\cdot L) \arrow[r, "\sim"] \arrow[d, "V\cdot L"] & \mathrm{hocolim}_{\mathbf{W_S}} DL \arrow[d, "L"] \\ \mathrm{hocolim}_{V\cdot \mathbf{W}} V\cdot D \arrow[r, "\sim"] & \mathrm{hocolim}_{\mathbf{W}} D \end{tikzcd}$$

that commutes up to homotopy and the horizontal maps are weak equivalences by Proposition 3. Now, by Theorem 2, the right vertical map is a weak equivalence and $V\cdot L$ pulls back homotopy colimits. □

For instance, with a bit of reflection, we see that the functor $U^+\cdot L : U^+\cdot \mathbf{W_S} \to U^+\cdot \mathbf{W}$ directly relates the poset forms of the coset geometries associated to the homotopy decompositions of Lemma 2 (appearing below) and Theorem 1 in the $I = \emptyset$ case. We also note a useful observation.

Proposition 4. *Define* $w(I)$ *as the longest* $v \in W_I$ *such that* $v \le w$ *in* $\mathbf{W}$. *The functor* $L_I : \mathbf{W} \to \mathbf{W}_I$ *given by* $w \mapsto w(I)$ *pulls back homotopy colimits. Moreover,* $L|_{L^{-1}[w,v]}$ *pulls back homotopy colimits for all intervals and* $V\cdot L_I$ *pulls back homotopy colimits for all* $V \le U^+$ *as in Corollary 3.*

Proof: Note that $w \downarrow L_I = L_I^{-1}(w(I) \downarrow \mathbf{W}_I) = w(I) \downarrow \mathbf{W}$ and $w(I) \downarrow \mathbf{W}$ has initial object $w(I)$. For closed intervals, $L_I^{-1}([w,v])$ still has initial object $w(I)$. The functor $V\cdot L_I$ is defined as in Corollary 3 with respect to the base $\mathbf{W}_I$, i.e. $\mathbf{W}_I$ is realized as the poset of subgroups $\{U_w\}_{w\in W_I}$. With this definition, $V\cdot L_I$ pulls back homotopy colimits for all $V \le U^+$ as in the proof Corollary 3. □

Proof of Theorem 2: By Proposition 2 it is sufficient to show that the geometric realization of the poset

$$\mathbf{X}^v_w := L^{-1}[v,w].$$

is contractible for all $v \le w \in \mathbf{W}$. Define $l[v,w]$ to be the maximal chain length in $[v,w]$. Let us proceed by induction on this length. If $v = w$, then wW_{I_w} is the (unique) terminal object of $\mathbf{X}^v_{v=w}$ and $|\mathbf{X}^v_{v=w}| \simeq \{*\}$.

Fix (v,w) with $v < w$. We will cover $\mathbf{X}^v_w$ as a category and apply Theorem 6 to show that $|\mathbf{X}^v_w| \simeq \{*\}$, inductively. Let us start to define the elements of this cover precisely.

For any $w \in \mathbf{W}$, there is a unique greatest $I_w \in \mathbf{S}$ such that w is longest in wW_{I_w}. By Theorem B, I_w is precisely the set of all i such that s_i is the right most letter of some reduced word expression of w. Define $\mathbf{Y}_w$ to be the full subcategory of the poset $\mathbf{W_S}$ with

$$Objects(\mathbf{Y}_w) = \{vW_J \in W_\mathbf{S} | vW_J \le wW_{I_w}\}.$$

Now, $\mathbf{X}^v_w$ is a full subcategory of $\mathbf{W_S}$ and

$$Objects(\mathbf{X}^v_w) \subseteq \bigcup_{v\le x\le w} Objects(\mathbf{Y}_x) \tag{33}$$

where $v \leq w$ refers to the weak Bruhat order on W (10). Because all chains in $\mathbf{W_S}$ are contained within some $\mathbf{Y}_x$, $\mathbf{X}^v_w$ is covered by $\{\mathbf{Y}_x \cap \mathbf{X}^v_w\}_{v\leq x\leq w}$, as a category (see Definition 2 in 2.5).

We now see that $\mathbf{X}^v_w$ is covered as a category by $\mathbf{X}^v_{ws_i}$ for all $i \in I_w$ and $\{\mathbf{X}^v_w \cap \mathbf{Y}_w\}$ since

$$\begin{aligned}\bigcup_{i\in I_w} Objects(\mathbf{X}^v_{ws_i}) &= \bigcup_{\substack{v\leq x\leq ws_i\\ i\in I_w}} Objects(\mathbf{Y}_x \cap \mathbf{X}^v_{ws_i})\\ &= \bigcup_{v\leq x<w} Objects(\mathbf{Y}_x \cap \mathbf{X}^v_w).\end{aligned}$$

and $\mathbf{X}^v_w$ is covered by $\{\mathbf{Y}_x \cap \mathbf{X}^v_w\}_{v\leq x\leq w}$ by (33).

Remove all empty elements of this cover. In particular, $\mathbf{X}^v_{ws_i}$ is non-empty only if $v \leq ws_i$. Define y to be the shortest element of wW_{I_w}. Any (non-empty) element of this cover contains the singleton coset $\{z := \sup\{v, y\}\}$ which exists and is less than or equal to w since $v < w$ and $y < w$. Thus, we can close this cover under intersection without introducing empty categories.

If $m \leq |I_w| + 1$ is the number of elements of this cover, we may define $\mathcal{U}$: $\Delta_{m+1} \to \mathbf{Cat}$ via $\{i_1, \dots i_k\} \mapsto \mathbf{U}_{i_1} \cap \dots \cap \mathbf{U}_{i_k}$ for some enumeration of this cover. Here Δ_{m+1} is the category of inclusions of facets in the standard $m+1$-simplex. By Theorem 6, hocolim$|\mathcal{U}|$ is weakly homotopy equivalent to $|\mathbf{X}^v_w|$. As $|\Delta_m| \simeq \{*\}$, it is enough to show that each $|\mathbf{U}_{i_1} \cap \dots \cap \mathbf{U}_{i_k}| \simeq \{*\}$ by induction on $l[v, w]$. In fact, we will show that each element of this cover is isomorphic to some $\mathbf{X}^{v_1}_{v_2}$ with $l[v_1, v_2] < l[v, w]$ or has a terminal object.

Case 1: $\bigcap_{i\in I\subseteq I_w} \mathbf{X}^v_{ws_i}$. Any non-empty intersection of $\mathbf{X}^v_{ws_i}$ for $i \in I_w$ is equal to some $\mathbf{X}^v_x$ with $x < w$ because any intersection of intervals $[v, ws_i]$ will be some interval $[v, x]$ and intersections pullback along functors. Inductively, $|\mathbf{X}^v_x| \simeq \{*\}$.

Case 2: $\bigcap_{i\in I\subseteq I_w} \mathbf{X}^v_{ws_i} \cap \mathbf{Y}_w$. Any intersection of $\mathbf{Y}_w$ and at least one non-empty $\mathbf{X}^v_{ws_i}$ with $i \in I_w$ will be equal to $\mathbf{X}^v_x \cap \mathbf{Y}_w$ for some $x < w$. Multiplication by y induces an isomorphism of posets

$$\mathbf{X}^{e_{I_w}}_{e_{I_w}} \xrightarrow{\simeq} \mathbf{Y}_w \tag{34}$$

for e_{I_w} the longest word in W_{I_w}. Now, $\mathbf{X}^v_x \cap \mathbf{Y}_w = \mathbf{X}^z_x \cap \mathbf{Y}_w$ for $z = \sup\{v, y\}$ and $\mathbf{X}^z_x \cap \mathbf{Y}_w$ is in bijection with $\mathbf{X}^{y^{-1}z}_{y^{-1}x}$ under (34). Observe that $l[y^{-1}z, y^{-1}x] = l[z, x] \leq l[v, x] < l[v, w]$ as $v \leq z \leq x < w$. By induction,

$$|\mathbf{X}^v_x \cap \mathbf{Y}_w = \mathbf{X}^z_x \cap \mathbf{Y}_w \cong \mathbf{X}^{y^{-1}z}_{y^{-1}x}| \simeq \{*\}.$$

Case 3: $\mathbf{X}^v_w \cap \mathbf{Y}_w$. In this case, there is terminal object, namely wW_{I_w}, and $|\mathbf{X}^v_w \cap \mathbf{Y}_w| \simeq \{*\}$. This completes the proof. □

3.2. **New homotopy decompositions for groups with RGD.** Theorem 1 will follow from Lemma 2 and Lemma 3 (below) by pulling back appropriate homotopy colimits.

Lemma 2. *Let $U^+ = U_\emptyset$ be the positive "unipotent" subgroup of a group with RGD and $\mathbf{W_S}$ be the poset underlying the Davis complex (see 2.6). Then the canonical map*

$$hocolim_{\mathbf{W_S}} BU_{w_J} \xrightarrow{\sim} BU^+ \tag{35}$$

induces a homotopy equivalence where w_J is the longest word in wW_J under the weak Bruhat order (10). More generally, if the standard parabolic subgroup P_I has a Levi decomposition then

$$hocolim_{\hat{U}^+_I\cdot\mathbf{W_S}} BU_{(\hat{u},w)} \xrightarrow{\sim} BU^+_I$$

where $\hat{U}_I^+ \cdot \mathbf{W_S}$ is the poset defined in Corollary 3 and each $U_{(\hat{u},w)}$ is isomorphic to a subgroup of some U_v.

Proof: Let us first show the $I = \emptyset$ case. We calculate

$$\begin{aligned} BU^+ &\simeq EU^+ \times_{U^+} \{*\} \simeq EU^+ \times_{U^+} \text{hocolim}_{\mathbf{S}} G/P_J^- \\ &\simeq \text{hocolim}_{\mathbf{S}} EU^+ \times_{U^+} G/P_J^- \\ &\simeq \text{hocolim}_{\mathbf{S}} \Big(\coprod_{w \in wW_J \in W/W_J} EU^+ \times_{U^+} U^+ wP_J^- \Big) \\ &\simeq \text{hocolim}_{\mathbf{W_S}} EU^+ \times_{U^+} U^+ wP_J^- \\ &\simeq \text{hocolim}_{\mathbf{W_S}} B(Stab_{U^+}\{wP_J^-\}) \end{aligned} \tag{36}$$

where the fourth equivalence requires the generalized Birkhoff decomposition (18) and the fifth uses the isomorphism of posets $wP_J^- \mapsto wW_J$ and Proposition 3 for $X : \mathbf{S} \to \mathbf{Sets}$ defined via $I \mapsto W/W_I$. Alternatively, it is not overly difficult to check directly that the map $uwP_J^- \mapsto (wW_J, uStab_{U^+}\{wP_J^-\})$ is an isomorphism of the poset forms of the coset geometries associated to $\text{hocolim}_{\mathbf{W_S}} B(Stab_{U^+}\{wP_J^-\})$ and $\text{hocolim}_{\mathbf{S}} BP_J^-$, respectively. Note that Lemma 1 identifies the stabilizer $Stab_{U^+}\{wP_J^-\}$ as U_{w_J}. This completes the $I = \emptyset$ case.

Whenever P_I has a Levi decomposition (23), $U_I^+ \rtimes \hat{U}_I^+ \cong U^+ \subset P_I^+ \cong U_I^+ \rtimes G_I$ so that

$$G = \coprod_{w \in W/W_J} U^+ wP_J^- = \coprod_{w \in W/W_J} U_I^+ \hat{U}_I^+ wP_J^-$$

by the generalized Birkhoff decomposition (18). Thus,

$$\begin{aligned} BU_I^+ &\simeq EU^+ \times_{U_I^+} \text{hocolim}_{\mathbf{S}} G/P_I^- \\ &\simeq \text{hocolim}_{\hat{U}_I^+ \cdot \mathbf{W_S}} B(Stab_{U_I^+}\{\hat{u}wP_I^-\}) \end{aligned}$$

as in (36). Here the map $u\hat{u}wP_J^- \mapsto (wW_J, uStab_{U_I^+}\{\hat{u}wP_I^-\})$ is an isomorphism of the poset forms of the coset geometries associated to $\text{hocolim}_{\hat{U}_I^+ \cdot \mathbf{W_S}} B(Stab_{U_I^+}\{\hat{u}wP_I^-\})$ and $\text{hocolim}_{\mathbf{S}} BP_J^-$, respectively, for $u \in U_I^+$ and $\hat{u} \in \hat{U}_I^+$.

Let us characterize $Stab_{U_I^+}\{\hat{u}wP_J^-\} = U_I^+ \cap \hat{u}wP_J^- w^{-1}\hat{u}^{-1}$. We see

$$\begin{aligned} \hat{u}^{-1}U_I^+\hat{u} \cap wP_J^- w^{-1} &= \hat{u}^{-1}U_I^+\hat{u} \cap U^+ \cap wP_J^- w^{-1} \\ &= \hat{u}^{-1}U_I^+\hat{u} \cap U_{wP_J^-}. \end{aligned} \tag{37}$$

Thus $Stab_{U_I^+}\{\hat{u}wP_J^-\} = \hat{u}U_{wP_J^-}\hat{u}^{-1} \cap U_I^+$. As J has finite type, each $\hat{u}U_{wP_J^-}\hat{u}^{-1} = \hat{u}U_{w_J}^-\hat{u}^{-1} \cong U_{w_J}$ by Lemma 1. This completes the proof. □

Recall that G_I (22) has a root group data structure with Weyl group W_I. The positive "unipotent" subgroup of this root group data structure is $\hat{U}_I^+$. Applying Theorem 2 to (35) gives the following.

Corollary 4. *Under the assumptions of Lemma 2, the canonical map induced by inclusions of subgroups*

$$hocolim_{w \in \mathbf{W}_I} BU_w \xrightarrow{\sim} B\hat{U}_I^+$$

is a homotopy equivalence where $\mathbf{W}_I$ is a poset under the weak Bruhat order (10).

We are now ready to prove our final lemma for Theorem 1.

Lemma 3. *The posets $\hat{U}_I^+ \cdot \mathbf{W_S}$ and $\hat{U}_I^+ \cdot \mathbf{W}$, defined in Corollary 3, have contractible geometric realizations.*

Proof: Recall the definition of $w(I)$ as the longest $v \in W_I$ such that $v \leq w$ in $\mathbf{W}$ and note that (17) implies $U_w \cap \hat{U}_I^+ = \bigcap_{v \in W_I} U_v \cap U_w = U_{w(I)}$. Thus, we have a commutative diagram of fibrations over BU_I^+ induced by inclusions of subgroups

$$\begin{tikzcd}
{|\hat{U}_I^+ \cdot \mathbf{W_S}|} \arrow[r] \arrow[d, "\hat{U}_I^+ \cdot L"'] & \mathrm{hocolim}_{\mathbf{W_S}} B(U_{w_J} \cap \hat{U}_I^+) \arrow[d, "L"'] \arrow[ddr, bend left] & \\
{|\hat{U}_I^+ \cdot \mathbf{W}|} \arrow[r] \arrow[d, "\hat{U}_I^+ \cdot L_I"'] & \mathrm{hocolim}_{\mathbf{W}} B(U_w \cap \hat{U}_I^+) \arrow[d, "L_I"'] \arrow[dr] & \\
{|\hat{U}_I^+ \cdot \mathbf{W}_I|} \arrow[r] & \mathrm{hocolim}_{\mathbf{W}_I} BU_{w(I)} \arrow[r] & B\hat{U}_I^+ \\
{|\hat{U}_I^+ \cdot (\mathbf{W}_I)_{\mathbf{S} \cap I}|} \arrow[r] \arrow[u, "\hat{U}_I^+ \cdot L"] & \mathrm{hocolim}_{(\mathbf{W}_I)_{\mathbf{S} \cap I}} BU_w \arrow[u, "L"] \arrow[ur] &
\end{tikzcd}$$

with the vertical maps induced by the indicated functors of index categories as defined in 3.1. In particular, the vertical maps are weak equivalences by Theorem 2, Corollary 3 and Proposition 4. The bottom fibration is simply the homotopy decomposition (35) of Lemma 2 for the (not necessarily compact) Levi factor G_I which carries a RGD structure with positive "unipotent" subgroup $\hat{U}_I^+$. Thus, we have $|\hat{U}_I^+ \cdot \mathbf{W_S}| \simeq |\hat{U}_I^+ \cdot \mathbf{W}| \simeq |\hat{U}_I^+ \cdot \mathbf{W}_I| \simeq \{*\}$ which completes the proof. $\square$

Proof of Theorem 1: By Lemma 2 and its proof all the statements of Theorem 1 follow from Theorem 2 and Corollary 3 except the claim that $|\hat{U}_I^+ \cdot \mathbf{W}|$ is contractible which is shown Lemma 3. Note that the facts that U_w are unipotent and Levi decompositions always in the Kac-Moody case are needed. $\square$

3.3. **Vanishing and simplification at $q \neq p$.** Our main application of the results of 3.2 is the following corollary.

Corollary 5. *Let $U_I^+(\mathbb{F})$ be the positive "unipotent" factor of parabolic subgroup of a discrete Kac-Moody group over a field of characteristic p. Then for any other prime q distinct from p, $H_n(BU_I^+(\mathbb{F}), \mathbb{F}_q) = 0$ for all $n > 0$.*

Proof: First consider $\mathbb{F} = \mathbb{F}_{p^k}$ for k fixed. Each $U_w(\mathbb{F}_{p^k})$ is a p group with $p^{kl(w)}$ elements where $l(w)$ is the reduced word length of w, c.f. (16). Thus, each $BU_w(\mathbb{F}_{p^k})$ is a $\mathbb{F}_q$-homology point. More generally, $BU_w(\mathbb{F})$ is a $\mathbb{F}_q$-homology point since each $U_w(\mathbb{F})$ has a normal series of length $l(w)$ with quotient groups isomorphic to $(\mathbb{F}, +)$. By the homology spectral sequence for homotopy colimits [7], $H_n(BU^+(\mathbb{F}_{p^k}), \mathbb{F}_q)$ is the homology of the poset $\mathbf{W_S}$ underlying the homotopy decomposition (35). Since the geometric realization of this poset is the contractible coset geometry of (31), $H_n(BU^+(\mathbb{F}), \mathbb{F}_q) = 0$ for all $n > 0$, completing the proof in the case of $I = \emptyset$. The same proof will work for all I since all $U_{(\hat{u},w)}(\mathbb{F})$ are unipotent and $|\hat{U}_I^+(\mathbb{F}) \cdot \mathbf{W_S}| \simeq \{*\}$ by Lemma 3. $\square$

An proof independent of the rank two case of Corollary 5 appears in [5].

The remainder of this section collects applications of Corollary 5, including the proof of Theorem 3.

Theorem 7. *Let $K(\mathbb{F})$ be a Kac-Moody group over a field of characteristic p with standard parabolic subgroups $P_I(\mathbb{F})$ and $G_I(\mathbb{F})$ the reductive, Lie Levi component*

subgroups for all $I \in \mathbf{S}$. The canonical map induced by inclusions of subgroups

$$\mathit{hocolim}_{I\in\mathbf{S}} BG_I(\mathbb{F}) \xrightarrow{\overset{q}{\sim}} BK(\mathbb{F}) \tag{38}$$

is a q-equivalence for any prime q distinct from p.

Proof: In this proof all groups mentioned are over a fixed $\mathbb{F}$. Consider the fibration sequence

$$BU_I \longrightarrow BP_I \longrightarrow BG_I \tag{39}$$

arising from the semidirect product decomposition $P_I \cong G_I \ltimes U_I$ (23). By Corollary 5, $BU_I(\mathbb{F})$ is a $\mathbb{F}_q$-homology point for all $I \in S$. The Serre spectral sequence for the fibration (39) shows $B(P_I \xrightarrow{\pi} G_I)$ induces a isomorphism on $\mathbb{F}_q$-homology. Note also that $G_I \xrightarrow{\iota} P_I \xrightarrow{\pi} G_I$ is the identity on G_I for ι the inclusion. Therefore, the map of graded abelian groups $H_*(B(G_I \xrightarrow{\iota} P_I), \mathbb{F}_q)$ must be the inverse of $H_*(B(P_I \xrightarrow{\pi} G_I), \mathbb{F}_q)$ by naturality. Recalling (32) the natural maps induced by inclusion of subgroups

$$\mathrm{hocolim}_{I\in S} BG_I \xrightarrow{\overset{q}{\sim}} \mathrm{hocolim}_{I\in S} BP_I \xrightarrow{\sim} BK$$

compose to yield the desired q-equivalence. □

Proof of Theorem 3: In the case at hand, Theorem 7 implies

$$\mathrm{hocolim}_{I\in S} BG_I(\overline{\mathbb{F}}_p) \xrightarrow{\overset{q}{\sim}} BK(\overline{\mathbb{F}}_p) \tag{40}$$

where $\overline{\mathbb{F}}_p$ is the algebraic closure of $\mathbb{F}_p$. Referring back to the construction Friedlander and Mislin used to produce Theorem A, the maps $BG_I(\overline{\mathbb{F}}_p) \to BG_I$ are induced by the zig-zag of groups

$$G_I(\overline{\mathbb{F}}_p) \longleftarrow G_I(Witt(\mathbb{F}_p)) \longrightarrow G_I(\mathbb{C}) \longrightarrow G_I \tag{41}$$

where $Witt(\mathbb{F}_p) \to \mathbb{C}$ is a *fixed* choice of embedding of the Witt vectors of $\mathbb{F}_p$ into $\mathbb{C}$ and $G_I(R)$ denotes the discrete algebraic group over R of the same type as the (topological) complex reductive Lie group G_I. Moreover, the maps of (41) are natural with respect to the maps of group functors $G_I(-) \hookrightarrow G_J(-)$. Taking classifying spaces, we have compatible maps

$$BG_I(\overline{\mathbb{F}}_p) \longleftarrow BG_I(Witt(\overline{\mathbb{F}}_p)) \longrightarrow BG_I(\mathbb{C}) \longrightarrow BG_I$$

which are all q-equivalences. Localizing at a prime q distinct from p, we obtain compatible $BG_I(\overline{\mathbb{F}}_p)^\wedge_q \overset{\overset{q}{\sim}}{\to} BG_I{}^\wedge_q$. Compatible $BG_I(\overline{\mathbb{F}}_p) \to BG_I$ are then produced via an arithmetic fibre square and induce a q-equivalence

$$\mathrm{hocolim}_{I\in S} BG_I(\overline{\mathbb{F}}_p) \overset{\overset{q}{\sim}}{\to} \mathrm{hocolim}_{I\in S} BG_I\ . \tag{42}$$

Thus, equations (2), (32), (38) and (42) induce

$$\begin{aligned} BK(\overline{\mathbb{F}}_p) \quad &\overset{\sim}{\leftarrow} \quad \mathrm{hocolim}_{I\in S} BP_I(\overline{\mathbb{F}}_p) \overset{\overset{q}{\sim}}{\rightleftarrows} \mathrm{hocolim}_{I\in S} BG_I(\overline{\mathbb{F}}_p) \\ &\overset{\overset{q}{\sim}}{\to} \quad \mathrm{hocolim}_{I\in S} BG_I \overset{\sim}{\to} BK. \end{aligned}$$

Choosing a fixed homotopy inverse for the arrow pointing to the left, we obtain the desired map. □

3.4. **Unstable Adams operations for Kac-Moody groups.** For $q \neq p$, we will construct a local unstable Adams operation $\psi^k : BK_q^\wedge \to BK_q^\wedge$ compatible with the Frobenius map. When W has no element of order p, we can assemble the local Adams maps via the arithmetic fibre square to obtain a global unstable Adams operation $\psi^k : BK \to BK$.

Proof of Theorem 4: Let us first construct $\psi^k : BK_q^\wedge \to BK_q^\wedge$ for $q \neq p$ by noting that, localizing of the map (4) constructed in Theorem 3 we have homotopy equivalences

$$(\mathrm{hocolim}_{I\in\mathbf{S}} BG_I(\overline{\mathbb{F}}_p)_q^\wedge)_q^\wedge \xrightarrow{\sim} (\mathrm{hocolim}_{I\in\mathbf{S}} BG_{I}{}_q^\wedge)_q^\wedge \xrightarrow{\sim} BK_q^\wedge$$

with the left equivalence induced by Theorem A. As in the proof of Theorem 3, we have compatible $BG_I(\overline{\mathbb{F}}_p)_q^\wedge \xrightarrow{\sim} BG_{I}{}_q^\wedge$ induced by the zig-zag of groups

$$G_I(\overline{\mathbb{F}}_p) \longleftarrow G_I(Witt(\mathbb{F}_p)) \longrightarrow G_I(\mathbb{C}) \longrightarrow G_I$$

where $Witt(\mathbb{F}_p) \to \mathbb{C}$ is a fixed embedding of the Witt vectors of $\mathbb{F}_p$ into $\mathbb{C}$ and $G_I(R)$ denotes the discrete algebraic group over R of the same type as the (topological) complex reductive Lie group G_I. From the naturality of the group functors, we have explicit topological models so that

$$\begin{array}{ccc} BG_I(\overline{\mathbb{F}}_p) & \xrightarrow{BG_I(\varphi^k)} & BG_I(\overline{\mathbb{F}}_p) \\ \Big\downarrow{\scriptstyle B(i)} & & \Big\downarrow{\scriptstyle B(i)} \\ BG_J(\overline{\mathbb{F}}_p) & \xrightarrow{BG_I(\varphi^k)} & BG_J(\overline{\mathbb{F}}_p) \end{array}$$

commutes for all $I \subset J$ in $\mathbf{S}$. Localizing, we have a map of diagrams and $\psi^k := (\varphi^k)_q^\wedge$ extends to $\mathrm{hocolim}_{I\in\mathbf{S}} BG_I(\overline{\mathbb{F}}_p)_q^\wedge \simeq BK_q^\wedge$.

When W has no elements of order p, we will work one prime at a time and assemble the maps using an arithmetic fibre square. We have already constructed $\psi^k : BK_q^\wedge \to BK_q^\wedge$ for $q \neq p$. At p, we have a homotopy equivalence

$$BN_p^\wedge \xrightarrow{\sim} BK_p^\wedge$$

where N is the normalizer of the maximal torus and this map is induced by the inclusion of groups [33]. Kumar [36, 6.1.8] presents N as being generated by T and $\{s_1, \dots s_n\}$ so that under the projection $\pi : N \to W$ s_i maps to a standard generator of W. Thus, we can attempt to define $\theta : N \to N$ in terms of these generators and need only check that Kumar's relations are satisfied to obtain a group homeomorphism. To get an unstable Adams operation we choose $t \mapsto t^{p^k}$ and $s_i \mapsto s_i$. Note that here p is odd and this is needed to verify Kumar's relations.

Because BK is a simply connected CW-complex [34], it is given as a homotopy pullback

$$\begin{array}{ccc} BK \simeq BK_{\mathbb{Z}}^\wedge & \xrightarrow{\prod(-)_q^\wedge} & \prod BK_q^\wedge \\ \Big\downarrow{\scriptstyle (-)_{\mathbb{Q}}^\wedge} & & \Big\downarrow{\scriptstyle (-)_{\mathbb{Q}}^\wedge} \\ BK_{\mathbb{Q}}^\wedge & \xrightarrow{\prod(-)_q^\wedge} & \prod(BK_q^\wedge)_{\mathbb{Q}}^\wedge \end{array} \tag{43}$$

known as the arithmetic fibre square where $(-)_{\mathbb{Z}}^\wedge$ and $(-)_{\mathbb{Q}}^\wedge$ denote localization with respect to $\mathbb{Z}$ and $\mathbb{Q}$ homology, respectively [15]. Now, by (2) we have homotopy equivalence

$$BK_{\mathbb{Q}}^\wedge \simeq (\mathrm{hocolim}_{I\in\mathbf{S}} BG_{I}{}_{\mathbb{Q}}^\wedge)_{\mathbb{Q}}^\wedge \simeq (\mathrm{hocolim}_{I\in\mathbf{S}} \prod K(2m_i, \mathbb{Q}))_{\mathbb{Q}}^\wedge \tag{44}$$

where the m_i vary for different BG_I and $K(n,\mathbb{Q})$ denotes the n^{th} Eilenberg-MacLane space [18]. The homomorphism $t \mapsto t^{p^k}$ of S^1 induces compatible self maps of the $K(2m_i,\mathbb{Q}) \simeq B^{2m_i-1}(S^1)^\wedge_{\mathbb{Q}}$ appearing in (44). We may now define the desired map with (43). $\square$

Remark 3. *In general, twisted Adams operations beyond those constructed here and coming from $Aut(\boldsymbol{S})$ are expected. See* [4] *for rank two examples.*

Now, that we have $\psi^k : BK^\wedge_q \to BK^\wedge_q$ let us note that when comparing $(BK^\wedge_q)^{h\psi^k}$ and $BK(\mathbb{F}_{p^k})^\wedge_q$ there is a natural map

$$BK(\mathbb{F}_{p^k})^\wedge_q \simeq (\mathrm{hocolim}_{I\in S} BG_I(\mathbb{F}_{p^k})^\wedge_q)^\wedge_q \longrightarrow (BK^\wedge_q)^{h\psi^k} \tag{45}$$

arising from the diagram $D : \mathbb{Z}\times\mathbf{S} \to \mathit{Spaces}$ via $(\bullet, W_I) \mapsto BG_I{}^\wedge_q$ on objects and $(n, W_I \hookrightarrow W_J) \mapsto (\psi^k)^n B(G_I \hookrightarrow G_J) = B(G_I \hookrightarrow G_J)(\psi^k)^n$ on morphisms. In particular, (45) is the localization of the canonical

$$\mathrm{hocolim}_{\mathbf{S}}(\mathrm{holim}_{\mathbb{Z}} BG_I(\mathbb{F}_{p^k})^\wedge_q) \longrightarrow \mathrm{holim}_{\mathbb{Z}}(\mathrm{hocolim}_{\mathbf{S}} BG_I(\mathbb{F}_{p^k})^\wedge_q). \tag{46}$$

Generally, we do not expect homotopy limits and colimits to commute. Our calculations in the next section that show that they very rarely do in rank two examples.

Question 1. *What is the structure of the homotopy fibre of (46)?*

4. Cohomology Calculations

Theorem 7 allows us to study $BK(\mathbb{F}_{p^k})$, the classifying spaces of a discrete Kac-Moody group over a finite field, in terms of *finite* reductive algebraic group classifying spaces, $BG_I(\mathbb{F}_{p^k})$ for $I \in \mathbf{S}$, after localizing at a prime q distinct from p. Furthermore, Theorem A reduces study of $BG_I(\mathbb{F}_{p^k})$ to understanding homotopy fixed points $(BG_I{}^\wedge_q)^{h\psi_I^k}$ under stable Adams operations, ψ_I^k. Thus, in principle, only the cohomology of compact Lie group classifying spaces is need as input data to compute $H^*(BK(\mathbb{F}_{p^k}), \mathbb{F}_q)$. In this section, we will begin such calculations and compare the results with $H^*((BK^\wedge_q)^{h\psi^k}, \mathbb{F}_q)$ for rank two, infinite dimensional Kac-Moody groups. We close with explicit calculations of $H^*(BU^+(\mathbb{F}_p, \mathbb{F}_p)$ in specific cases where $W \cong (\mathbb{Z}/2\mathbb{Z})^{*n}$.

4.1. Restriction to Rank 2.

To compare $(BK^\wedge_q)^{h\psi^k}$ and $BK(\mathbb{F}_{p^k})^\wedge_q$, we will partially compute their $\mathbb{F}_q$-cohomology rings. These computations will become more tractable by restricting to rank 2 Kac-Moody groups and q odd. Notably $H^*(BK, \mathbb{F}_q)$ and its ψ^k-action can be determined explicitly. In the rank 2, non-Lie, case $BK(\mathbb{F}_{p^k})^\wedge_q \simeq \mathrm{hocolim}_{I\in\mathbf{S}}(BG_I{}^\wedge_q)^{h\psi_I^k}$ is simply a homotopy pushout. To compute cohomology, we turn to the Mayer-Vietoris sequence. We will also restrict to q odd, so that for $I \in \mathbf{S}$

$$H^*(BG_I, \mathbb{F}_q) \cong H^*(BT, \mathbb{F}_q)^{W_I} \tag{47}$$

with the restriction map from $H^*(BG_I, \mathbb{F}_q)$ to $H^*(BT, \mathbb{F}_q)$ inducing this isomorphism [18]. The determination of $H^*((BG_I{}^\wedge_q)^{h\psi^k}, \mathbb{F}_q)$ for $I \in \mathbf{S}$ will occur in 4.2; this subsection will investigate $H^*((BK^\wedge_q)^{h\psi^k}, \mathbb{F}_q)$.

In the rank 2 case, we have [3, 33]

$$H^*(BK, \mathbb{F}_q) = \mathbb{F}_q[x_4, x_{2l}] \otimes \Lambda(x_{2l+1}) \tag{48}$$

TABLE 1. The $\mathbb{F}_q$-algebra structure for $E_2^{*,*}$ converging to $H^*((BK_q^\wedge)^{h\psi^k}, \mathbb{F}_q)$ with bidegrees $|x_i| = (-1, i+1)$, $|\gamma_n(x_i)| = (-n, (i+1)n)$, and $|s_i| = (0, i)$.

	$p^{2k} = 1(mod\ q)$	$p^{2k} \neq 1(mod\ q)$
$p^{kl} = 1(mod\ q)$	$\Lambda(x_3, x_{2l-1}) \otimes \Gamma(x_{2l}) \otimes \mathbb{F}_q[s_4, s_{2l}] \otimes \Lambda(s_{2l+1})$	$\Lambda(x_{2l-1}) \otimes \Gamma(x_{2l}) \otimes \mathbb{F}_q[s_{2l}] \otimes \Lambda(s_{2l+1})$
$p^{kl} \neq 1(mod\ q)$	$\Lambda(x_3) \otimes \mathbb{F}_q[s_4]$	$\mathbb{F}_q$

as a ring where Λ denotes an exterior algebra and $l := l(\{a, b\}, q) \geq 2$ is a positive integer depending on q and the generalized Cartan matrix for K, i.e. some non-singular 2×2 matrix given by

$$\begin{bmatrix} 2 & -a \\ -b & 2 \end{bmatrix} \tag{49}$$

for a and b positive integers such that $ab \geq 4$. An explicit description of l is given in [33]. Work in [3] will determine the map ψ^k induces on $\mathbb{F}_q$-cohomology.

Proposition 5. *For K a rank 2, infinite dimensional complex Kac-Moody group and q odd, ψ^k acts on $H^*(BK, \mathbb{F}_q)$ (48) via $(x_4, x_{2l}, x_{2l+1}) \mapsto (p^{2k}x_4, p^{lk}x_{2l}, p^{lk}x_{2l+1})$.*

Proof: Because $H^*(BG_I, \mathbb{F}_q)$ is concentrated in even degrees for $I \in \mathbf{S}$, the Mayer-Vietoris sequence associated to the homotopy pushout presentation of BK (2) reduces to an exact sequence

$$\begin{aligned} 0 \to H^{2*}(BK, \mathbb{F}_q) &\to H^*(BG_1, \mathbb{F}_q) \oplus H^*(BG_2, \mathbb{F}_q) \to H^*(BT, \mathbb{F}_q) \\ &\to H^{2*+1}(BK, \mathbb{F}_q) \to 0 \end{aligned} \tag{50}$$

where $H^{2*}(BK, \mathbb{F}_q) = H^*(BG_1, \mathbb{F}_q) \cap H^*(BG_2, \mathbb{F}_q) = \mathbb{F}_q[x_4, x_{2l}]$ and $H^{2*+1}(BK, \mathbb{F}_q) = \langle x_{2l+1} \rangle H^{2*}(BK, \mathbb{F}_q)$ for x_{2l+1} the image of a homogeneous degree $2l$ class under the connecting homomorphism [3]. The k^{th} unstable Adams operation ψ^k acts on $H^*(BT, \mathbb{F}_q)$ via multiplication by p^k on generators, and commutes with the restriction (47) $H^*(BG_I, \mathbb{F}_q) \to H^*(BT, \mathbb{F}_q)$ [43]. $\square$

We are presently unable to fully compute $H^*((BK_q^\wedge)^{h\psi^k}, \mathbb{F}_q)$ for all rank 2 Kac-Moody groups. However, there is sufficient information at $E_2^{*,*}$ in the Eilenberg-Moore spectral sequence associated to $H^*((BK_q^\wedge)^{h\psi^k}, \mathbb{F}_q)$ to determine that in most rank 2 cases $H^*(BK(\mathbb{F}_{p^k}), \mathbb{F}_q) \neq H^*((BK_q^\wedge)^{h\psi^k}, \mathbb{F}_q)$, in contrast to the Lie case.

Theorem 8. *Consider the Eilenberg-Moore spectral sequence (EMSS) for the homotopy pullback of the diagram*

$$\begin{tikzcd} & BK_q^\wedge \arrow[d, "\Delta"] \\ BK_q^\wedge \arrow[r, "1\times\psi^k\circ\Delta"] & (BK \times BK)_q^\wedge \end{tikzcd} \tag{51}$$

converging to $H^*((BK_q^\wedge)^{h\psi^k}, \mathbb{F}_q)$ where ψ^k is the k^{th} unstable Adams operation. For q odd, $E_2^{*,*}$, as a $\mathbb{F}_q$-algebra, is given in Table 1.

Proof: The $E_2^{*,*}$-page for the cohomological EMSS for (51) is given by

$$Tor^{*,*}_{H^*(BK,\mathbb{F}_q)\otimes H^*(BK,\mathbb{F}_q)}(H^*(BK, \mathbb{F}_q), H^*(BK, \mathbb{F}_q)) \tag{52}$$

$$\cong Tor^{*,*}_{\mathbb{F}_q[x_4,y_4,x_{2l},y_{2l}]\otimes\Lambda(x_{2l+1},y_{2l+1})}(\mathbb{F}_q[s_4, s_{2l}] \otimes \Lambda(s_{2l+1}), \mathbb{F}_q[t_4, t_{2l}] \otimes \Lambda(t_{2l+1}))$$

TABLE 2. $E_2^{*,*}$ converging to $H^*((BK_q^\wedge)^{h\psi^k}, \mathbb{F}_q)$ in terms of $Tor^{*,*}$ where $z_i = y_i - x_i$ for y_i and x_i.

$(mod\ q)$	$E_2^{*,*}$
$p^{kl} = 1$ and $p^{2k} = 1$	$Tor^{*,*}_{\mathbb{F}_q[z_4, z_{2l}] \otimes \Lambda(z_{2l+1})}(\mathbb{F}_q[s_4, s_{2l}] \otimes \Lambda(s_{2l+1}), \mathbb{F}_q)$
$p^{kl} \neq 1$ and $p^{2k} = 1$	$Tor^{*,*}_{\mathbb{F}_q[z_{2l}] \otimes \Lambda(z_{2l+1})}(\mathbb{F}_q[s_{2l}] \otimes \Lambda(s_{2l+1}), \mathbb{F}_q)$
$p^{kl} = 1$ and $p^{2k} \neq 1$	$Tor^{*,*}_{\mathbb{F}_q[z_4]}(\mathbb{F}_q[s_4], \mathbb{F}_q)$
$p^{kl} \neq 1$ and $p^{2k} \neq 1$	$Tor^{*,*}_{\mathbb{F}_q}(\mathbb{F}_q, \mathbb{F}_q)$

Here left copy of $H^*(BK, \mathbb{F}_q)$ a $H^*(BK, \mathbb{F}_q) \otimes H^*(BK, \mathbb{F}_q)$-module via $1 \times \psi^k \circ \Delta$ which is given by the ring homomorphism $(y_4, y_{2l}, y_{2l+1}) \mapsto (p^{2k}s_4, p^{lk}s_{2l}, p^{lk}s_{2l+1})$ and $x_i \mapsto t_i$ on generators by Proposition 5. Of course, the $H^*(BK, \mathbb{F}_q) \otimes H^*(BK, \mathbb{F}_q)$-module structure on the right $H^*(BK, \mathbb{F}_q)$ is given by $y_i, x_i \mapsto t_i$.

To simplify $E_2^{*,*}$ we will employ a change of ring isomorphism.

Proposition A ([38] p. 280). *Let A, B, and C be k-algebras over a field k. Then,*

$$Tor^n_A(M, N) = Tor^n_C(N, L) = 0 \tag{53}$$

for all $n > 0$ implies

$$Tor^n_{A \otimes B}(M, N \otimes_C L) \cong Tor^n_{B \otimes C}(M \otimes_A N, L) \tag{54}$$

where M, N and L have the appropriate module structures so that (54) is well defined.

Let us use this change of ring isomorphisms with

$$\begin{aligned}(A, B, C) &= (\mathbb{F}_q[x_4, x_{2l}] \otimes \Lambda(x_{2l+1}), \mathbb{F}_q[y_4 - x_4, y_{2l} - x_{2l}] \otimes \Lambda(y_{2l+1} - x_{2l+1}), \mathbb{F}_q), \\ (M, N, L) &= (\mathbb{F}_q[s_4, s_{2l}] \otimes \Lambda(s_{2l+1}), \mathbb{F}_q[t_4, t_{2l}] \otimes \Lambda(t_{2l+1}, \mathbb{F}_q).\end{aligned}$$

This gives

$$E_2^{*,*} \cong Tor^{*,*}_{\mathbb{F}_q[z_4, z_{2l}] \otimes \Lambda(z_{2l+1})}(\mathbb{F}_q[s_4, s_{2l}] \otimes \Lambda(s_{2l+1}), \mathbb{F}_q) \tag{55}$$

where $z_i = y_i - x_i$ acts via $(z_4, z_{2l}, z_{2l-1}) \mapsto ((p^{2k}-1)s_4, (p^{lk}-1)s_{2l}, (p^{lk}-1)s_{2l+1})$. We may employ Proposition A further if $p^{2k} - 1$ or $p^{lk} - 1$ is nonzero modulo q. In this way, we obtain Table 2.

For example, the $p^{kl} = 1(mod\ q)$ and $p^{2k} \neq 1(mod\ q)$ case is obtained via the choices

$$(A, B, C, M, N, L) = (\mathbb{F}_q, \mathbb{F}_q[z_{2l}] \otimes \Lambda(z_{2l+1}), \mathbb{F}_q[z_4], \mathbb{F}_q[s_{2l}] \otimes \Lambda(s_{2l+1}), \mathbb{F}_q[s_4], \mathbb{F}_q)$$

and the other entries are obtained similarly. In Table 2, all the modules are trivial over their respective $\mathbb{F}_q$-algebras.

For q odd, all the $\mathbb{F}_q$-algebras in Table 2 are finitely generated free graded commutative. Let $\mathbb{F}_q(X)$ denote a free graded commutative algebra on the graded set X.

We follow [38, p. 258-260] to compute $E_2^{*,*}$. If one resolves the trivial $\mathbb{F}_q(X)$-module $\mathbb{F}_q$ using the Koszul complex $\Omega(X)$, then

$$Tor^{*,*}_{\mathbb{F}_q(X)}(L, \mathbb{F}_q) = H(\Omega(X) \otimes L, d_L) \tag{56}$$

where L is any $\mathbb{F}_q(X)$-module. In particular, $\Omega(X)$ is $\Lambda(s^{-1}X^{even}) \otimes \Gamma(s^{-1}X^{odd})$ where Γ denotes the divided power algebra. Here $s^{-1}X^{even}$ is a set of odd degree generators obtained from the even degree elements of X by decreasing the their degree by one and $s^{-1}X^{odd}$ is defined analogously. In all the cases at hand, L is a trivial $\mathbb{F}_q(X)$-module which implies d_L is zero. Thus, $E_2^{*,*}$ is given by $\Omega(X) \otimes L$

TABLE 3. $H^*((BG_1{}^\wedge_q)^{h\psi^k}, \mathbb{F}_q) \cong H^*((BG_2{}^\wedge_q)^{h\psi^k}, \mathbb{F}_q)$ for rank 2 Kac-Moody groups of infinite type and q odd.

	$H^*((BG_1{}^\wedge_q)^{h\psi^k}, \mathbb{F}_q) \cong H^*((BG_2{}^\wedge_q)^{h\psi^k}, \mathbb{F}_q)$
$p^k = 1(mod\ q)$	$\Lambda(z_1, z_3) \otimes \mathbb{F}_q[s_2, s_4]$
$p^k = -1(mod\ q)$	$\Lambda(z_3) \otimes \mathbb{F}_q[s_4]$
$p^k \neq \pm 1(mod\ q)$	$\mathbb{F}_q$

TABLE 4. $E_2^{*,*}$ collapsing to $H^*((BG_1{}^\wedge_q)^{h\psi^k}, \mathbb{F}_q) \cong H^*((BG_2{}^\wedge_q)^{h\psi^k}, \mathbb{F}_q)$ for $z_i = x_i - x_i'$ the difference of the generators for two copies of $H^*((BG_1{}^\wedge_q)^{h\psi^k}, \mathbb{F}_q) \cong H^*((BG_2{}^\wedge_q)^{h\psi^k}, \mathbb{F}_q)$.

	$E_2^{*,*}$
$p^k = 1(mod\ q)$	$Tor^{*,*}_{\mathbb{F}_q[z_2,z_4]}(\mathbb{F}_q[s_2, s_4], \mathbb{F}_q)$
$p^k = -1(mod\ q)$	$Tor^{*,*}_{\mathbb{F}_q[z_4]}(\mathbb{F}_q[s_4], \mathbb{F}_q)$
$p^k \neq \pm 1(mod\ q)$	$Tor^{*,*}_{\mathbb{F}_q}(\mathbb{F}_q, \mathbb{F}_q)$

(see Table 1). As the EMSS is a spectral sequence of $\mathbb{F}_q$-algebras [38], we have computed $E_2^{*,*}$ as an algebra. □

Note that from Table 1 the EMSS computing $H^*((BK^\wedge_q)^{h\psi^k}, \mathbb{F}_q)$ collapses at $E_2^{*,*}$ if $p^{kl} \neq 1(mod\ q)$ for degree reasons. However, if $p^{kl} = 1(mod\ q)$, $d_r(\gamma_r(x_{2l}))$ is potentially nonzero.

4.2. **Levi component calculations and a proof Theorem 5.** Note that for odd q and $I = \{i\}$

$$H^*(BG_I, \mathbb{F}_q) \cong H^*(BT, \mathbb{F}_q)^{r_i} = \mathbb{F}_q[p_2(x,y), p_4(x,y)] = \mathbb{F}_q[s_2, s_4]$$

with r_i generating W_I and $p_j(x,y)$ homogenous polynomials of degree j in the degree 2 generators of $H^*(BT, \mathbb{F}_q)$, i.e. $p_2(x,y)$ is linear in x and y. The action of ψ^k is given by a simple scalar multiplication on generators and we can compute $H^*((BG_I{}^\wedge_q)^{h\psi^k}, \mathbb{F}_q)$ for $I \in S = \{\emptyset, \{1\}, \{2\}\}$.

Theorem 9. *The Eilenberg-Moore spectral sequence (EMSS) for the homotopy pullback computes $H^*((BG_I{}^\wedge_q)^{h\psi^k}, \mathbb{F}_q)$ where ψ^k is the k^{th} unstable Adams operation. For q odd, we may compute $H^*((BG_I{}^\wedge_q)^{h\psi^k}, \mathbb{F}_q)$ for $I \in S = \{\emptyset, \{1\}, \{2\}\}$ and, up to isomorphism, it is given in Table 3 and*

$$\begin{aligned} H^*((BT^\wedge_q)^{h\psi^k}, \mathbb{F}_q) = \Lambda(z_1, z_1') \otimes \mathbb{F}_q[s_2, s_2'] &\iff p^k = 1(mod\ q) \text{ and} \\ \mathbb{F}_q &\iff p^k \neq 1(mod\ q)\ . \end{aligned} \tag{57}$$

Proof: Let us first compute $H^*((BG_1{}^\wedge_q)^{h\psi^k}, \mathbb{F}_q) \cong H^*((BG_2{}^\wedge_q)^{h\psi^k}, \mathbb{F}_q)$. If we employ the techniques in the proof of Theorem 8, we obtain Table 4 in which all $\mathbb{F}_q(X)$-modules are trivial. Kozsul resolutions give Table 3, but here the spectral sequence collapses at $E_2^{*,*}$ for degree reasons. Equation (57) is obtained similarly. □

Proof Theorem 5: Let us set notation for the Mayer-Vietoris sequence for computing $H^*(\text{hocolim}_{\mathbf{S}}(BG_{I_q}^{\wedge})^{h\psi_I^k}, \mathbb{F}_q)$

$$\begin{aligned} \cdots \overset{\partial_{n-1}}{\to} H^n(\text{hocolim}_{\mathbf{S}}(BG_{I_q}^{\wedge})^{h\psi_I^k}, \mathbb{F}_q) &\overset{\rho_n}{\to} H^n((BG_{1_q}^{\wedge})^{h\psi^k}, \mathbb{F}_q) \oplus H^n((BG_{2_q}^{\wedge})^{h\psi^k}, \mathbb{F}_q) \\ &\overset{\Delta_n}{\to} H^n((BT_q^{\wedge})^{h\psi^k}, \mathbb{F}_q) \overset{\partial_n}{\to} \cdots . \end{aligned} \tag{58}$$

We will proceed by cases.

Case 1: $p^k \neq 1(mod\ q)$. Here the vanishing of $H^*((BT_q^{\wedge})^{h\psi^k}, \mathbb{F}_q)$ gives the second and third rows of Table 5 by (57-58). If $p^k \neq \pm 1(mod\ q)$, then the theorem clearly holds. If instead $p^k = -1(mod\ q)$, then comparing Tables 1 and 5 in degrees 3 and 5 gives $H^*((BK_q^{\wedge})^{h\psi^k}, \mathbb{F}_q) \neq H^*(\text{hocolim}_{I\in\mathbf{S}}(BG_{I_q}^{\wedge})^{h\psi^k}, \mathbb{F}_q)$.

Case 2: $p^k = 1(mod\ q)$. In this case, $H^*((BG_{I_q}^{\wedge})^{h\psi^k}, \mathbb{F}_q)$ is isomorphic to the $\mathbb{F}_q$-cohomology of free loops on BG_I all for $I \in \mathbf{S}$ [32]. Furthermore, we know Δ from the Mayer-Vietoris sequence (58) almost completely by loc. cit. A careful count of dimensions will show $H^{4l}(\text{hocolim}_{I\in\mathbf{S}}(BG_I)^{h\psi^k})_q^{\wedge}, \mathbb{F}_q) \neq H^{4l}((BK_q^{\wedge})^{h\psi^k}, \mathbb{F}_q)$. Let $|\cdot|$ denote the rank of a vector space for this purpose.

On $E_2^{*,*}$ of the EMSS for $H^*((BK_q^{\wedge})^{h\psi^k}, \mathbb{F}_q)$ total degree $4l$ has rank 6 for l odd and rank 8 for l even. Because this is a spectral sequence of algebras, d_r of all these generators vanish since they are products of permanent cocycles for degree reasons noting that $q \neq 2$ implies $\gamma_2(x_{2l}) = \frac{\gamma_1(x_{2l})}{2}^2$. Indeed, they are all permanent cocycles as they are not the target of any differential for degree reasons and we recover $|H^{4l}((BK_q^{\wedge})^{h\psi^k}, \mathbb{F}_q)|$.

Considering (57) and Table 3 we see that

$$|H^{4l-1}((BT_q^{\wedge})^{h\psi^k}, \mathbb{F}_q))| = 2 \cdot |(BG_i{}^{\wedge_q})^{h\psi^k}, \mathbb{F}_q)| = 4l$$

for $i \in \{1,2\}$. It follows that

$$|H^{4l}(\text{hocolim}_{I\in\mathbf{S}}(BG_{I_q}^{\wedge})^{h\psi_I^k}, \mathbb{F}_q)| = |ker(\Delta_{4l-1})| + |ker(\Delta_{4l})|$$

by elementary homological methods. By [3, 32], there are isomorphisms of graded abelian groups

$$ker(\Delta) \cong H^*((BT_q^{\wedge})^{h\psi^k}, \mathbb{F}_q)^W \cong \mathbb{F}_q[s_4, s_{2l}] \otimes \Lambda(x_3, x_{2l-1})$$

where W is the infinite dihedral Weyl group of K which acts on $H^*((BT_q^{\wedge})^{h\psi^k}, \mathbb{F}_q) \cong \Lambda(x_1, x_1') \otimes \mathbb{F}_q[s_2, s_2']$. Here we extend the standard action on $H^*(BT, \mathbb{F}_q) \cong \mathbb{F}_q[s_2, s_2']$ via the identification $ds_i = x_i$ so that d commutes with the action and satisfies the Leibnitz rule. In the terminology of [32], $(BT_q^{\wedge})^{h\psi^k}$ is a twisted loop space with associated derivation d and d commutes with restriction. Considering $\mathbb{F}_q[s_4, s_{2l}] \otimes \Lambda(x_3, x_{2l-1})$ in degree $4l$ and $4l-1$ gives that $H^{4l}(\text{hocolim}_{I\in\mathbf{S}}(BG_I)^{h\psi^k})_q^{\wedge}, \mathbb{F}_q)$ has rank 5 for l odd and rank 6 for l even. Hence $p^k = 1(mod\ q)$ implies $H^*((BK_q^{\wedge})^{h\psi^k}, \mathbb{F}_q)$ and $H^*(\text{hocolim}_{I\in\mathbf{S}}(BG_{I_q}^{\wedge})^{h\psi^k}, \mathbb{F}_q)$ are distinct (and nontrivial) for odd q. $\square$

Table 5 summarizes our deductions about the structure of

$$H^*(\text{hocolim}_{\mathbf{S}}(BG_{I_q}^{\wedge})^{h\psi_I^k}, \mathbb{F}_q) \cong H^*(BK(\mathbb{F}_{p^k}), \mathbb{F}_q)$$

within the proof of Theorem 5. The methods of [3], where (50) is used to compute $H^*(BK, \mathbb{F}_q)$, and a good understanding of the derivation associated to a twisted loop space should in principle allow further computation in the $p^k = 1(mod\ q)$ case. We preform no such calculations here but note that [5, Theorem 8.3] computes $H^*(BK(\mathbb{F}_{p^k}), \mathbb{F}_q)$ in many of these interesting cases.

TABLE 5. $H^*(\text{hocolim}_{\mathbf{S}}(BG_I{}^{\wedge}_q)^{h\psi_I^k}, \mathbb{F}_q)$ for rank 2 Kac-Moody groups of infinite type.

	$H^*(\text{hocolim}_{\mathbf{S}}(BG_I{}^{\wedge}_q)^{h\psi_I^k}, \mathbb{F}_q)$
$p^k = 1(mod\ q)$	$H^*(\text{hocolim}_{\mathbf{S}} LBG_I, \mathbb{F}_q)$
$p^k = -1(mod\ q)$	$\Lambda(z_3) \otimes \mathbb{F}_q[s_4] \oplus \Lambda(z_3') \otimes \mathbb{F}_q[s_4']$
$p^k \neq \pm 1(mod\ q)$	$\mathbb{F}_q$

4.3. **Examples of $H^*(BU^+(\mathbb{F}_{p^k}), \mathbb{F}_p)$.** In this section, we preform some preliminary calculations for the group cohomology of $U^+(\mathbb{F}_{p^k})$ at p using our new homotopy decomposition (3) and briefly discuss considerations in the general case. In our examples, we will see that $H^*(BU^+(\mathbb{F}_{p^k}), \mathbb{F}_p)$ is infinitely generated as a ring. Along the way, we give explicit descriptions of the groups $U^+(R)$ underlying our cohomology calculations. Unless otherwise stated, when we refer to Kac-Moody groups in this subsection we mean some image of Tits's [46] explicit (discrete, minimal, split) Kac-Moody group functor $K(-)$: **Rings** $\to$ **Groups** from the category of commutative rings with unit to the category of groups. Likewise, $U^+(R)$ will be Tits's subgroup generated by the positive (real) root groups (14). When R is clear from context, we will simply write K or U^+.

Our simplest (and most explicit) calculations will be for $U_2 := U_2^+(R) \leq K_2(R)$ the positive "unipotent" subgroup of the infinite dimensional Kac-Moody group $K_2(R)$ with generalized Cartan matrix

$$\begin{bmatrix} 2 & -a \\ -b & 2 \end{bmatrix} \tag{59}$$

where $a, b \geq 2$. In this case the relevant diagram of unipotent subgroups of $U_2^+(R)$ indexed over W is

$$\ldots U_{tst} \longleftarrow U_{st} \longleftarrow U_t \longleftarrow e \longrightarrow U_s \longrightarrow U_{st} \longrightarrow U_{sts} \ldots \tag{60}$$

as here W is the infinite dihedral group. When $a = b = 2$ and R is a field, we have an affine Kac-Moody group $K_2(R) \cong GL_2(R[t, t^{-1}])$ and U_2^+ is known explicitly (c.f. [45]). We will identify $U_w(R)$ directly from the presentation of Tits [46] and recover $U_2^+(R)$ from the colimit presentation induced by (3). In fact, for any Kac-Moody group with generalized Cartan matrix $A = (a_{ij})_{1 \leq i,j \leq k}$ such that $|a_{ij}| \geq 2$ for all $1 \leq i, j \leq k$ the finite dimensional unipotent subgroups $U_w(R)$ have a very simple form.

Lemma 4. *Let $K(R)$ be the discrete Kac-Moody group over R with generalized Cartan matrix $A = (a_{ij})_{1 \leq i,j \leq k}$ such that $|a_{ij}| \geq 2$ for all $1 \leq i, j \leq k$. Then, for $U_w(R) \leq K(R)$ defined by (15) we have*

$$U_w(R) \cong (R, +)^{\oplus l(w)}$$

such that the image of $U_w(R) \hookrightarrow U_{ws}(R)$ for $l(w) + 1 = l(ws)$ is the first $l(w)$ factors.

Proof: Recall that $w^{-1}\Phi^+ \cap \Phi^- = \Theta_w := \{\alpha_{i_1}, \alpha_{i_2}, \ldots, ws_{i_l}\alpha_{i_l}\}$ and that the multiplication map

$$U_{\alpha_{i_1}} \times U_{s_{i_1}\alpha_{i_2}} \times \ldots \times U_{ws_{i_l}\alpha_{i_l}} \xrightarrow{m} U_w$$

is an isomorphism of sets (16). It is immediate from the presentation of Tits [46] that for $\alpha, \beta \in \Phi^+$, $\{\mathbb{N}\alpha + \mathbb{N}\beta\} \cap \Phi^+ = \{\alpha, \beta\}$ implies $\langle U_\alpha, U_\beta \rangle = U_\alpha \times U_\beta$. Moreover, the W of action on Φ guarantees that $\{\mathbb{N}\alpha + \mathbb{N}\beta\} \cap \Phi^+ \subset \Theta_w$ whenever $\alpha, \beta \in \Theta_w$. Direct computation in the hypothesized cases of the action of W on the simple

roots (13) shows that each of the positive coefficients of $ws_{i_l}\alpha_{i_l}$ expressed in terms of the simple roots is maximal and one coefficient is strictly largest within Θ_w. In particular, if $\alpha \in \Theta_w$ is distinct from $ws_{i_l}\alpha_{i_l}$ then U_α and $U_{ws_{i_l}\alpha_{i_l}}$ commute because the coefficients demand that if $m_1\alpha + m_2 ws_{i_l}\alpha_{i_l} \in \Theta_w$, then (m_1, m_2) is $(0,1)$ or $(m_1, 0)$. However, $m_1\alpha \in \Phi^+ \implies m_1 = 1$ [36].

Since $u, v \in U_w$ implies

$$uv = (u_1u_2 \dots u_{l-1})u_l(v_1v_2 \dots v_{l-1})v_l = (u_1u_2 \dots u_{l-1})(v_1v_2 \dots v_{l-1})u_lv_l$$

for $u_j, v_j \in U_{s_{i_1} \dots s_{i_{j-1}}\alpha_j}$, we have $U_w \cong U_{ws_{i_l}} \times U_\gamma$ for $\gamma = ws_{i_l}\alpha_{i_l}$. The lemma follows by induction on $l(w)$ and the fact that all root subgroups are isomorphic to $(R, +)$. □

Returning to $U_2(R)$ and considering the right telescope of (60), we obtain a projective limit of graded abelian groups upon taking group cohomology

$$(61)\quad H^*(e, \mathbb{F}_p) \twoheadleftarrow H^*(U_s, \mathbb{F}_p) \twoheadleftarrow H^*(U_{st}, \mathbb{F}_p) \twoheadleftarrow H^*(U_{sts}, \mathbb{F}_p) \dots$$

which is a sequence of surjections by the Kunneth theorem. In particular,

$$\varprojlim{}^1 H^*(U_w(R), \mathbb{F}_p) = 0 \tag{62}$$

and the cohomology spectral sequence of the mapping telescope of classifying spaces [38] collapses at $E_2^{*,*}$.

There is some subtly in defining the positive "unipotent" subgroup of a Kac-Moody group over an arbitrary ring (c.f. Remark 3.10 (d) in [46]) so we will state the rest of our results in this section only over fields.

Theorem 10. *Let $U_2 := U_2^+(L) \le K_2(L)$ be the positive "unipotent" subgroup of the Kac-Moody group $K_2(L)$ over a field L with generalized Cartan matrix given by (59). Then we may compute the group cohomology of $U_2(L)$ as*

$$H^*(BU_2(L), \mathbb{F}) = \bigotimes_{\mathbb{Z}>0} H^*(L, \mathbb{F}) \oplus \bigotimes_{\mathbb{Z}>0} H^*(L, \mathbb{F})$$

where L is considered as an abelian group and $\mathbb{F}$ is a field.

Proof: As $\lim^1_{\mathbf{W}^{op}} H^*(U_w(L), \mathbb{F}) = 0$ in this case (62), the spectral sequence of the homotopy colimit of classifying spaces collapses at $E_2^{*,*}$ and computes $H^*(BU^+(L), \mathbb{F})$ as the projective limit of graded abelian groups $\lim_{\mathbf{W}^{op}} H^*(U_w(L), \mathbb{F})$. Working one telescope at a time, we see the limit of graded abelian groups (61) is just $\lim_{\mathbb{Z}>0} H^*((L)^n, \mathbb{F})$ by Lemma 4. This limit is computed in each gradation so that, unlike the limit of underlying abelian groups, only finite tensor words are possible in each gradation. The left telescope is computed in the same way. As the double telescope is the one point union of the left and right telescopes, $H^*(BU_2(L), \mathbb{F})$ is simply the direct sum of their $\mathbb{F}$-cohomologies. □

For the case of interest, $H^*(BU^+(\mathbb{F}_{p^k}), \mathbb{F}_p)$, cohomology classes are represented by finite tensor words in the generators of $H^*(\mathbb{F}_{p^k}, \mathbb{F}_p)$. For instance, for $k = 1$ we have

$$H^*((\mathbb{Z}/p\mathbb{Z})^n, \mathbb{F}_p) = \begin{cases} \mathbb{F}_2[x_1, \dots, x_n] & p = 2 \\ \Lambda[x_1, \dots, x_n] \otimes \mathbb{F}_p[y_1, \dots, y_n] & p \text{ odd} \end{cases}$$

where $|x_i| = 1$ and $|y_i| = 2$. Applying the double telescope construction

$$\begin{aligned} H^*(BU^+(\mathbb{Z}/2\mathbb{Z}), \mathbb{F}_2) &= (\varprojlim \mathbb{F}_2[x_1, \dots, x_n])^{\oplus 2} \\ &= \mathbb{F}_2[x_1, , \dots, x_n, \dots] \oplus \mathbb{F}_2[\overline{x}_1, \dots, \overline{x}_n, \dots]. \end{aligned}$$

For p odd

$$\begin{aligned} H^*(BU^+(\mathbb{Z}/p\mathbb{Z}), \mathbb{F}_p) &= (\lim_{\leftarrow} \Lambda[x_1, \ldots, x_n] \otimes \mathbb{F}_p[y_1, \ldots, y_n])^{\oplus 2} \\ &= \Lambda[x_1, \ldots, x_n, \ldots] \otimes \mathbb{F}_p[y_1, \ldots, y_n, \ldots] \\ &\oplus \Lambda[\overline{x}_1, \ldots, \overline{x}_n, \ldots] \otimes \mathbb{F}_p[\overline{y}_1, \ldots, \overline{y}_n, \ldots]. \end{aligned}$$

Lemma 4 lets us determine the group structure of more general $U^+(R)$ which have an interesting presentation. Briefly, when Lemma 4 applies we have

$$U^+(R) = \mathrm{colim}_{w \in \mathbf{W}} R^{l(w)}$$

where all maps are inclusions $R^{l(v)} \hookrightarrow R^{l(w)}$ with image the first $l(v)$ factors. This leads to the presentation

$$U^+(R) = \langle U_\alpha | u_\alpha u_\beta u_\alpha^{-1} u_\beta^{-1} = 1, \text{ if } \exists w \text{ with } w\{\alpha, \beta\} \subset \Phi^- \rangle$$

where $w \in W$, u_γ is an arbitrary element of U_γ and $\gamma \in \Phi^+$. Pictorially, there is a generating $U_\alpha \cong (R, +)$ for each node in Hasse diagram of the poset W, i.e. the directed graph with vertices and edges $(W, \{(w, ws) | l(w) + 1 = l(ws)\})$ (e.g. see Figure 1). Generating groups U_α and U_β commute exactly when there is a (unique) path between them in this directed graph. If there is no path between the corresponding nodes, $U_\alpha * U_\beta$ embeds into U^+.

For example, if $K(R)$ has rank 3 with generalized Cartan matrix such as

$$\begin{bmatrix} 2 & -2 & -3 \\ -2 & 2 & -2 \\ -2 & -4 & 2 \end{bmatrix}, \tag{63}$$

then the poset $\mathbf{W}$ is generated by the directed valence-3 tree pictured in Figure 1. Two elements commute exactly when they are both contained in some $U_w(R)$.

Starting with the homotopy decomposition (3) of Theorem 1, it is straight forward to apply Theorem 6 and Theorem D to observe

$$\mathrm{colim}_{w \in \mathbf{W}} R^{l(w)} \cong \mathrm{colim}_{m \in \mathbb{N}} (\mathrm{colim}_{\mathbf{W}_m} R^{l(w)})$$

for $\mathbb{N}$ the linearly ordered poset and $\mathbf{W}_m$ is the full subposet of $\mathbf{W}$ of elements of length at most m since W is a tree. Set $U_{\mathbf{W}_m} := \mathrm{colim}_{\mathbf{W}_m} U_w$. By Corollary 2 we have a commutative diagram

$$\begin{tikzcd} \mathrm{hocolim}_{L^{-1}(\mathbf{W}_m)} U_{\mathbf{W}_m}/R^{l(w_I)} \arrow[r] \arrow[d] & \mathrm{hocolim}_{L^{-1}(\mathbf{W}_m)} B(R^{l(w_I)}) \arrow[d, "L"] \arrow[dr] & \\ \mathrm{hocolim}_{\mathbf{W}_m} U_{\mathbf{W}_m}/R^{l(w)} \arrow[r] & \mathrm{hocolim}_{\mathbf{W}_m} B(R^{l(w)}) \arrow[r] & BU_{\mathbf{W}_m} \end{tikzcd}$$

whose vertical maps are weak equivalences. Since $\mathrm{hocolim}_{L^{-1}(\mathbf{W}_m)} U_{\mathbf{W}_m}/R^{l(w_I)}$ can be modeled by a 1-dimensional CW-complex and is simply connected by Corollary 1, it must be contractible so that

$$B(\mathrm{colim}_{\mathbf{W}_m} R^{l(w)}) \simeq (\mathrm{hocolim}_{\mathbf{W}_m} B(R^{l(w)})). \tag{64}$$

Thus, we have homotopy equivalences

$$\begin{aligned} BU^+(R) &\simeq \mathrm{hocolim}_{w \in \mathbf{W}} B(R^{l(w)}) \\ &\simeq \mathrm{hocolim}_{\mathbb{N}} (\mathrm{hocolim}_{\mathbf{W}_m} B(R^{l(w)})) \\ &\simeq \mathrm{hocolim}_{\mathbb{N}} B(\mathrm{colim}_{\mathbf{W}_m} R^{l(w)}) \end{aligned} \tag{65}$$

induced by the obvious maps. Homology and cohomology calculations for $B(\mathrm{colim}_{\mathbf{W}_m} R^{l(w)}) \simeq (\mathrm{hocolim}_{\mathbf{W}_m} B(R^{l(w)}))$ can be preformed by iterated Mayer-Vietoris calculations. For instance, for K with the generalized Cartan matrix (63)

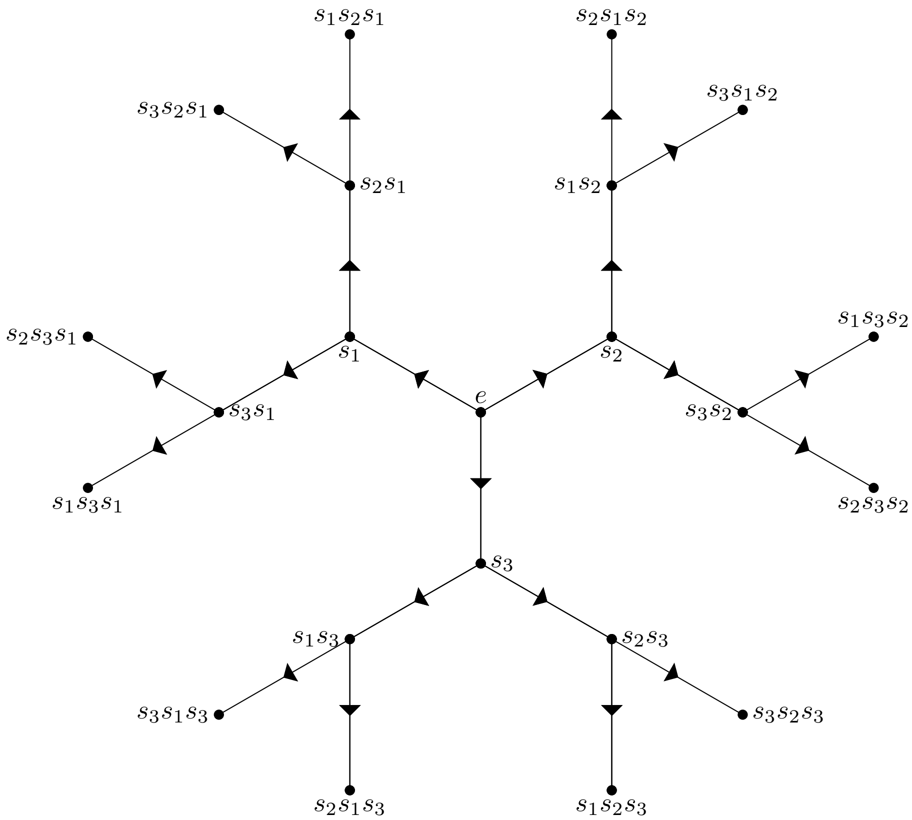

FIGURE 1. A picture of the graph underlying the poset $\mathbf{W} = \langle s_1, s_2, s_3 | s_i^2 = 1\rangle$ and the weak Bruhat order with elements of length at most 3 labeled.

considered above we have

$$\begin{aligned} \operatorname{colim}_{\mathbf{W}_m} R^{l(w)} &= [R^m *_{R^{m-1}} R^m *_{R^{m-2}} R^m \dots R^m *_{R^2} R^m \\ &*_k \quad R^m *_{R^2} R^m *_{R^3} R^m \dots R^m *_{R^{m-1}} R^m]^{*3} \end{aligned} \tag{66}$$

and each R^i of the amalgamated products denotes the first i factors. For example, this implies

$$H^*(\operatorname{colim}_{\mathbf{W}_3} (\mathbb{Z}/2\mathbb{Z})^{l(w)}, \mathbb{F}_2) = (\mathbb{F}_2[x, x_0, x_1, x_{00}, x_{01}, x_{10}, x_{11}]/I)^{\oplus 3}$$

when $m = 3$ where

$$I = \langle x_0x_1, x_0\{x_{10}, x_{11}\}, x_1\{x_{00}, x_{01}\}, x_{00}\{x_{10}, x_{10}, x_{11}\}, x_{01}\{x_{10}, x_{11}\}, x_{10}x_{11}\rangle$$

as nonidentity nodes in the directed graph underlying $\mathbf{W}_3$ (and pictured in Figure 1) contribute generators and the multiplicative structure is determined by whether the corresponding root groups commute or generate a free product. The three direct summands correspond to the each of the three main branches of the tree underlying $\mathbf{W}_3$. For instance, on the s_2 branch nodes correspond to generators via $s_2 \leftrightarrow x$, $s_1s_2 \leftrightarrow x_0$, $s_3s_2 \leftrightarrow x_1$, $s_2s_1s_2 \leftrightarrow x_{00}$, $s_3s_1s_2 \leftrightarrow x_{01}$, $s_1s_3s_2 \leftrightarrow x_{10}$, and $s_2s_3s_2 \leftrightarrow x_{11}$. The other branches have a similar correspondence. This cohomology may be obtained inductively from the expression of $\operatorname{colim}_{\mathbf{W}_3} R^{l(w)}$ given in (66). More generally, this process gives in the following result.

Theorem 11. *Let $K(L)$ be a discrete Kac-Moody over a field L with generalized Cartan matrix $A = (a_{ij})_{1\leq i,j\leq n}$ such that $|a_{ij}| \geq 2$ for all $1 \leq i, j \leq n$ with positive*

"unipotent" subgroup $U^+(L)$. Then the cohomology of $BU_+(L)$ with coefficients in any field $\mathbb{F}$ is given as

$$\begin{aligned} H^*(BU^+(L),\mathbb{F}) &= \lim_{\leftarrow} H^*(hocolim_{\mathbf{W}_m} B(L^{l(w)}),\mathbb{F}) \\ &= \otimes_{\mathbf{W}-\{e\}} H^*(L,\mathbb{F})/I \end{aligned}$$

where the ideal I is generated by all products of pairs of elements in different copies of $H^*(L,\mathbb{F})$ corresponding to nonidentity nodes in the directed graph underlying the poset $\mathbf{W}$ that are disconnected.

For the rank 3 example discussed above (63), $H^*(BU^+(\mathbb{F}_2),\mathbb{F}_2)$ is the $\mathbb{F}_2$-algebra generated by the nodes (except the root) of an infinitely extended Figure 1 with relations $xy = 0$ for generators x and y whose nodes are not connected.

All the examples described here concern Kac-Moody groups with Weyl groups whose Hasse diagrams are trees so that (64-66) is available to clarify inductive arguments and diagrams over W_m can be seen as iterated pushouts. Because the cohomology spectral sequence calculation is trivialized for limits satisfying the Mittag-Leffler condition, we pose the following question.

Question 2. *Let $K = K(\mathbb{F}_{p^k})$ be a discrete minimal Kac-Moody group over $\mathbb{F}_{p^k}$ with unipotent p-subgroups $U_w(\mathbb{F}_{p^k})$ defined by (15). When does the diagram of group cohomologies D_K : $\boldsymbol{W} \to$ **Groups** $\to$ **Graded Abelian Groups** given by $w \mapsto U_w(\mathbb{F}_{p^k}) \mapsto H^*(U_w(\mathbb{F}_{p^k}),\mathbb{F}_p)$ satisfy the Mittag-Leffler condition?*

In the tree shaped cases not covered here, a positive answer to this question would permit cohomology calculations as outlined in this section. Obviously, Lemma 4 greatly simplifies our calculations. For example, for an infinite dimensional Kac-Moody group with generalized Cartan matrix

$$\begin{bmatrix} 2 & -a \\ -1 & 2 \end{bmatrix}$$

for $a \geq 4$, it is a priori possible that the 3-dimensional unipotent group U_{sts} is a Heisenberg group (see [5] for details on Φ^+ in this case).

References

[1] Peter Abramenko and Kenneth S. Brown, *Buildings*, Graduate Texts in Mathematics, vol. 248, Springer, New York, 2008. Theory and applications.

[2] Jaume Aguadé, *p-compact groups as subgroups of maximal rank of Kac-Moody groups*, J. Math. Kyoto Univ. **49** (2009), no. 1, 83–112.

[3] Jaume Aguadé, Carles Broto, Nitu Kitchloo, and Laia Saumell, *Cohomology of classifying spaces of central quotients of rank two Kac-Moody groups*, J. Math. Kyoto Univ. **45** (2005), no. 3, 449–488.

[4] Jaume Aguadé and Albert Ruiz, *Maps between classifying spaces of Kac-Moody groups*, Adv. Math. **178** (2003), no. 1, 66–98.

[5] ———, *Cohomology of Kac-Moody groups over a finite field*, Preprint (2012), pp. 24, available at `arXiv:1206.1352v3[math.AT]`.

[6] A. K. Bousfield, *The localization of spaces with respect to homology*, Topology **14** (1975), 133–150.

[7] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics, Vol. 304, Springer-Verlag, Berlin, 1972.

[8] Carles Broto and Nitu Kitchloo, *Classifying spaces of Kac-Moody groups*, Math. Z. **240** (2002), no. 3, 621–649.

[9] Pierre-Emmanuel Caprace, *"Abstract" homomorphisms of split Kac-Moody groups*, Mem. Amer. Math. Soc. **198** (2009), no. 924, xvi+84.

[10] Pierre-Emmanuel Caprace and Bertrand Rémy, *Groups with a root group datum*, Innov. Incidence Geom. **9** (2009), 5–77.

[11] Ruth Charney and Michael W. Davis, *The $K(\pi,1)$-problem for hyperplane complements associated to infinite reflection groups*, J. Amer. Math. Soc. **8** (1995), no. 3, 597–627.

[12] Michael W. Davis, *The geometry and topology of Coxeter groups*, London Mathematical Society Monographs Series, vol. 32, Princeton University Press, Princeton, NJ, 2008.

[13] Tom De Medts, Ralf Gramlich, and Max Horn, *Iwasawa decompositions of split Kac-Moody groups*, J. Lie Theory **19** (2009), no. 2, 311–337.

[14] Emmanuel Dror Farjoun, *Fundamental group of homotopy colimits*, Adv. Math. **182** (2004), no. 1, 1–27.

[15] E. Dror, W. G. Dwyer, and D. M. Kan, *An arithmetic square for virtually nilpotent spaces*, Illinois J. Math. **21** (1977), no. 2.

[16] Daniel Dugger and Daniel C. Isaksen, *Topological hypercovers and $\mathbb{A}^1$-realizations*, Math. Z. **246** (2004), no. 4, 667–689.

[17] W. G. Dwyer and D. M. Kan, *A classification theorem for diagrams of simplicial sets*, Topology **23** (1984), no. 2, 139–155.

[18] W. G. Dwyer and C. W. Wilkerson, *The elementary geometric structure of compact Lie groups*, Bull. London Math. Soc. **30** (1998), no. 4, 337–364.

[19] Mikhail Ershov, *Golod-Shafarevich groups with property (T) and Kac-Moody groups*, Duke Math. J. **145** (2008), no. 2, 309–339.

[20] John D. Foley, *Comparing Kac-Moody groups over the complex numbers and fields of positive characteristic*, PhD Thesis (2012).

[21] Eric M. Friedlander, *Étale homotopy of simplicial schemes*, Annals of Mathematics Studies, vol. 104, Princeton University Press, Princeton, N.J., 1982.

[22] Eric M. Friedlander and Guido Mislin, *Cohomology of classifying spaces of complex Lie groups and related discrete groups*, Comment. Math. Helv. **59** (1984), no. 3, 347–361.

[23] Stéphane Gaussent and Guy Rousseau, *Kac-Moody groups, hovels and Littelmann paths*, Ann. Inst. Fourier (Grenoble) **58** (2008), no. 7, 2605–2657 (English, with English and French summaries).

[24] Paul G. Goerss and John F. Jardine, *Simplicial homotopy theory*, Progress in Mathematics, vol. 174, Birkhäuser Verlag, Basel, 1999.

[25] Ralf Gramlich and Andreas Mars, *Isomorphisms of unitary forms of Kac-Moody groups over finite fields*, J. Algebra **322** (2009), no. 2, 554–561.

[26] André Haefliger, *Complexes of groups and orbihedra*, Group theory from a geometrical viewpoint (Trieste, 1990), World Sci. Publ., River Edge, NJ, 1991, pp. 504–540.

[27] Jesper Grodal, *The Classification of p-Compact Groups and Homotopical Group Theory*, Proceedings of the International Congress of Mathematicians, Vol. 1, Hyderabad, India, 2010.

[28] J. Hollender and R. M. Vogt, *Modules of topological spaces, applications to homotopy limits and E_∞ structures*, Arch. Math. (Basel) **59** (1992), no. 2, 115–129.

[29] Victor G. Kac, *Constructing groups associated to infinite-dimensional Lie algebras*, Infinite-dimensional groups with applications (Berkeley, Calif., 1984), Math. Sci. Res. Inst. Publ., vol. 4, Springer, New York, 1985, pp. 167–216.

[30] V. G. Kac and D. H. Peterson, *Defining relations of certain infinite-dimensional groups*, Astérisque **Numero Hors Serie** (1985), 165–208. The mathematical heritage of Élie Cartan (Lyon, 1984).

[31] Victor G. Kac and Dale H. Peterson, *On geometric invariant theory for infinite-dimensional groups*, Algebraic groups Utrecht 1986, Lecture Notes in Math., vol. 1271, Springer, Berlin, 1987, pp. 109–142.

[32] Daisuke Kishimoto and Akira Kono, *On the cohomology of free and twisted loop spaces*, J. Pure Appl. Algebra **214** (2010), no. 5, 646–653.

[33] Nitu Kitchloo, *Topology of Kac-Moody groups*, MIT Thesis (1998).

[34] ———, *On the topology of Kac-Moody groups*, Manuscript (2008).

[35] ———, *Dominant K-theory and integrable highest weight representations of Kac-Moody groups*, Adv. Math. **221** (2009), no. 4, 1191–1226.

[36] Shrawan Kumar, *Kac-Moody groups, their flag varieties and representation theory*, Progress in Mathematics, vol. 204, Birkhäuser Boston Inc., Boston, MA, 2002.

[37] Bertram Kostant and Shrawan Kumar, *The nil Hecke ring and cohomology of G/P for a Kac-Moody group G*, Adv. in Math. **62** (1986), no. 3, 187–237.

[38] John McCleary, *A user's guide to spectral sequences*, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 58, Cambridge University Press, Cambridge, 2001.

[39] Gabor Moussong, *Hyperbolic Coxeter Groups*, PhD Thesis (1998).

[40] Daniel Quillen, *Higher algebraic K-theory. I*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 85–147. Lecture Notes in Math., Vol. 341.

[41] Stephen A. Mitchell, *Quillen's theorem on buildings and the loops on a symmetric space*, Enseign. Math. (2) **34** (1988), no. 1-2, 123–166.

[42] Bertrand Rémy, *Kac-Moody groups as discrete groups*, Essays in geometric group theory, Ramanujan Math. Soc. Lect. Notes Ser., vol. 9, Ramanujan Math. Soc., Mysore, 2009, pp. 105–124.

[43] Stefan Jackowski, James McClure, and Bob Oliver, *Homotopy classification of self-maps of BG via G-actions. I*, Ann. of Math. (2) **135** (1992), no. 1, 183–226.

[44] R. W. Thomason, *Homotopy colimits in the category of small categories*, Math. Proc. Cambridge Philos. Soc. **85** (1979), no. 1, 91–109.

[45] Jacques Tits, *Ensembles ordonnés, immeubles et sommes amalgamées*, Bull. Soc. Math. Belg. Sér. A **38** (1986), 367–387 (1987) (French).

[46] ———, *Uniqueness and presentation of Kac-Moody groups over fields*, J. Algebra **105** (1987), no. 2, 542–573.

[47] Volkmar Welker, Günter M. Ziegler, and Rade T. Živaljević, *Homotopy colimits—comparison lemmas for combinatorial applications*, J. Reine Angew. Math. **509** (1999), 117–149.

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